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Null Killing vectors and reductions of the self-duality equations

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Abstract. We find all null conformal Killing vectors in R^4 with the metric $\mathrm{d}t\,\mathrm{d}u - \mathrm{d}x\,\mathrm{d}y$. We classify one and two-dimensional totally null algebras generated by such vectors. Reductions of the self-dual Yang–Mills equations are obtained for gauge fields invariant with respect to these algebras.

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1. Introduction

Let $R^{2,2}$ denote the manifold R^4 with global coordinates $x^\mu = (t, u, x, y)$ and the metric

$$\mathrm{d}s^2 = \mathrm{d}t\,\mathrm{d}u - \mathrm{d}x\,\mathrm{d}y \tag{1.1}$$

of the signature $++--$. The Yang–Mills field on $R^{2,2}$ is represented by a one-form $A = A_\mu\,\mathrm{d}x^\mu$ with values in the Lie algebra of a Lie group G. The form A undergoes the following gauge transformations

$$A \to A' = g^{-1}Ag + g^{-1}\,\mathrm{d}g \tag{1.2}$$

where $g(x^\mu) \in G$. The field strengths of A are given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\nu + [A_\mu, A_\nu]. \tag{1.3}$$

We say that A is self-dual iff $F_{\mu\nu}$ is Hodge self-dual with respect to the metric (1.1). For some choice of orientation this condition amounts to the equations

$$F_{ux} = 0, \qquad F_{ty} = 0, \qquad F_{tu} + F_{xy} = 0. \tag{1.4}$$

Equations (1.4) are integrability conditions of the following linear equations [W1, BZ] for a matrix function $\psi(\lambda, x^\mu)$, where $\det(\psi) \neq 0$ and λ is a parameter,

$$(\partial_u + A_u - \lambda(\partial_y + A_y))\psi = 0, \tag{1.5}$$

$$(\partial_x + A_x - \lambda(\partial_t + A_t))\psi = 0. \tag{1.6}$$

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The system of equations (1.4) is an important example of so-called completely integrable equations (see e.g. [AC]). The point symmetry group of (1.4) is the conformal group $C(2.2)$ (isomorphic to $O(3,3)$). Many lower-dimensional completely integrable equations, including Korteweg de Vries (KdV) and Nonlinear Schrödinger (NLS) equations [MS], can be obtained from (1.4) (or its complexification) by assuming an invariance of A under a subgroup of $C(2,2)$ (for reviews, see [W2, AC, MW]). Not all reductions of this type are known, mainly because there is no complete classification of subgroups of the conformal group [Wi] (see [KLG] and [DS] for partial results in the Euclidean or complex case).

In this paper we consider algebras of vector fields on $R^{2,2}$ generated by one or two (orthogonal) null conformal Killing vectors K_i. Such algebras (we call them 'totally null') correspond to particular subgroups of $C(2,2)$. An invariance of A under one of these subgroups is equivalent to the equation(s)

$$L_{K_i} A = 0, \tag{1.7}$$

where L_{K_i} denotes the Lie derivative along K_i. We classify all the totally null algebras modulo conformal transformations (section 2) and we find reductions of the self-duality equations for gauge fields satisfying (1.7) (section 3). Some of these reductions are already known [W3, S, T] (see also [St]).

Each of the two-dimensional algebrae generated by K_1, K_2 defines at each point of $R^{2,2}$ a totally null plane which is either self-dual (then the tensor $K_{1[\mu}K_{2\nu]}$ satisfies equations (1.4)) or antiself-dual (then $K_{1[\mu}K_{2\nu]}$ satisfies equations following from (1.4) by the interchange of x and y). In the case of self-dual planes the reduction of the self-dual Yang–Mills equations is singular in the sense that the linear system (1.5), (1.6) does not reduce to a lower-dimensional one (one cannot assume an invariance of ψ with respect to K_i [T]). We are not able to prove that, in this case, equations (1.4) are trivial (as partial differential equations (PDE's)) for any gauge group, however, there are indications (section 3) that this is the case.

2. Null conformal Killing vectors

The conformal transformations of $R^{2,2}$ are generated by translations, rotations, dilations and the special conformal transformation

$$x'^{\mu} = s^{-2} x^{\mu}, \tag{2.1}$$

where $s^2 = tu - xy$. Every one-dimensional subgroup of $C(2,2)$ generates a conformal Killing vector field K on $R^{2,2}$ which preserves (1.1) up to a factor $a(x)$

$$L_K g = ag. \tag{2.2}$$

It is easy to write down the general expression for K. However, it is not easy to split the set of vector fields K into classes of fields which are related by conformal transformations of coordinates (this problem is equivalent to finding all conjugacy classes of elements of the Lie algebra of $C(2,2)$ [DPWZ]). We show in the following theorem that such classification is quite simple for null Killing vectors, i.e. under the assumption that

$$K = K^t\partial_t + K^u\partial_u + K^x\partial_x + K^y\partial_y, \qquad K^tK^u - K^xK^y = 0. \tag{2.3}$$

Theorem 1. *All null conformal Killing vectors in $R^{2,2}$ are given by*

$$K = (a_1u + a_2y + a_3)[(b_1u + b_2x + b_3)\partial_u + (b_1y + b_2t + b_4)\partial_y] \\ +(a_1x + a_2t + a_4)[(b_1u + b_2x + b_3)\partial_x + (b_1y + b_2t + b_4)\partial_t], \tag{2.4}$$

where a_i, b_i are real constants defined modulo the transformation $a_i \to ca_i$, $b_i \to c^{-1}b_i$, $c \in R$. Every vector (2.4) can be put into one of the inequivalent forms

$$K = \partial_u, \tag{2.5}$$

$$K = a(u\partial_u + y\partial_y), \qquad a = \text{ constant} \tag{2.6}$$

by means of a conformal transformation which preserves an orientation.

Proof. Any vector satisfying (2.3) can be written as

$$K = \alpha(\partial_y + \mu\partial_t + \nu\partial_u + \mu\nu\partial_x) \tag{2.7}$$

or

$$K = K^x\partial_x + K^t\partial_t \tag{2.8}$$

or

$$K = K^x\partial_x + K^u\partial_u. \tag{2.9}$$

Expressions (2.8) and (2.9) can be obtained from (2.7) (with $\mu = 0$ or $\nu = 0$) by simple interchanges of coordinates which preserve (1.1). For this reason we will focus on the case (2.7). In this case the Killing equation (2.2) gives rise to the following equations

$$\partial_x\alpha = 0, \qquad \partial_u(\alpha\mu) = 0, \tag{2.10}$$

$$\partial_t(\alpha\nu) = 0, \qquad \partial_y(\alpha\mu\nu) = 0, \tag{2.11}$$

$$\partial_x(\alpha\mu) - \partial_u\alpha = 0, \qquad \partial_y(\alpha\nu) - \partial_t(\alpha\mu\nu) = 0, \tag{2.12}$$

$$\partial_y(\alpha\mu) - \partial_u(\alpha\mu\nu) = 0, \qquad \partial_x(\alpha\nu) - \partial_t\alpha = 0, \tag{2.13}$$

$$\partial_t(\alpha\mu) + \partial_u(\alpha\nu) - \partial_x(\alpha\mu\nu) - \partial_y\alpha = 0. \tag{2.14}$$

For $\mu = 0$ or $\nu = 0$ equations (2.10)–(2.14) yield, respectively,

$$K = (b_1u + b_2x + b_3)\partial_u + (b_1y + b_2t + b_4)\partial_y, \tag{2.15}$$

$$K = (a_1u + a_2y + a_3)\partial_y + (a_1x + a_2t + a_4)\partial_t. \tag{2.16}$$

Expressions (2.15) and (2.16) are special cases of (2.4).

If $\alpha\mu\nu \neq 0$ then equation (2.14) follows from equations (2.10)–(2.13). Equations (2.12) and (2.13) can be replaced by

$$\partial_u\mu + \mu\partial_x\mu = 0, \qquad \partial_y\mu + \mu\partial_t\mu = 0, \tag{2.17}$$

$$\partial_y\nu + \nu\partial_u\nu = 0, \qquad \partial_t\nu + \nu\partial_x\nu = 0, \tag{2.18}$$

(for instance, the first equation in (2.17) follows, in virtue of (2.10), from the first equation in (2.12)). A direct consequence of (2.17) and (2.18) is that the functions μ and ν satisfy the wave equation

$$\partial_t\partial_u\mu - \partial_x\partial_y\mu = 0, \qquad \partial_t\partial_u\nu - \partial_x\partial_y\nu = 0. \tag{2.19}$$

Equations (2.10) and (2.11) can be written as the following system of differential equations for the function α

$$\partial_x \log\alpha = 0, \qquad \partial_u \log\alpha = \partial_x\mu, \tag{2.20}$$

$$\partial_t \log\alpha = \partial_x\nu, \qquad \partial_y \log\alpha = \partial_t\mu + \partial_u\nu. \tag{2.21}$$

Integrability conditions of (2.20), (2.21) are linear equations for μ and ν, which yield, in virtue of (2.19),

$$\mu = \mu_1 + t\mu_2 + x\mu_3, \qquad \nu = \nu_1 + u\nu_2 + x\nu_3, \tag{2.22}$$

where $\mu_i = \mu_i(u, y)$ and $\nu_i = \nu_i(t, y)$. Substituting (2.22) into (2.17) and (2.18) yields

$$\mu = \frac{a_1 x + a_2 t + a_4}{a_1 u + a_2 y + a_3}, \qquad \nu = \frac{b_1 u + b_2 x + b_3}{b_1 y + b_2 t + b_4}, \tag{2.23}$$

where a_i and b_i are constants. Given (2.23) one can solve equations (2.20), (2.21) for α. The corresponding vector field (2.7) takes the form (2.4).

In order to prove that every field (2.4) can be transformed into (2.5) or (2.6) we will consider separately fields for which exactly n of the constants a_1, a_2, b_1, b_2 vanish. If $n = 0$ then the rescaling $t' = a_2 b_2 t,\ u' = a_1 b_1 u,\ x' = a_1 b_2 x,\ y' = a_2 b_1 y$ transforms all constants a_1, a_2, b_1, b_2 to identity. In the new coordinates K takes the form (hereafter we will omit primes after every transformation of coordinates)

$$K = (u + y + a_3)[(u + x + b_3)\partial_u + (y + t + b_4)\partial_y] \\ +(x + t + a_4)[(u + x + b_3)\partial_x + (y + t + b_4)\partial_t]. \tag{2.24}$$

In virtue of the translation

$$t' = t + b_4, \qquad u' = u + a_3, \qquad x' = x + b_3 - a_3, \qquad y' = y \tag{2.25}$$

followed by the special conformal transformation (2.1) K transforms into

$$K = (1 + au)(\partial_x - \partial_u) + (1 - ay)(\partial_y - \partial_t), \tag{2.26}$$

where $a = a_4$. For $a \neq 0$ a shift in u and y yields

$$K = au(\partial_x - \partial_u) + ay(\partial_t - \partial_y), \tag{2.27}$$

and expression which transforms into (2.6) by means of the transformation $t' = t + y$, $u' = u,\ x' = x + u,\ y' = y$. For $a = 0$ equation (2.26) yields

$$K = \partial_x - \partial_u + \partial_y - \partial_t, \tag{2.28}$$

hence (2.5) follows by means of the transformation $t = -t' - u' - x' - y'$, $u = -u'$, $x = u' + x',\ y = u' + y'$.

If exactly one of the constants a_1, a_2, b_1, b_2 vanishes we can achieve $a_2 = 0$, $a_1 b_1 b_2 \neq 0$ by a simple interchange of coordinates. The constants a_1, b_1, b_2 can be set equal to identity due to the transformation $t' = b_2 t,\ u' = a_1 b_1 u,\ x' = a_1 b_2 x,\ y' = b_1 y$. Transformation (2.25) followed by (2.1) yield

$$K = au(\partial_x - \partial_u) + (1 - ay)(\partial_y - \partial_t), \qquad a = \text{constant}. \tag{2.29}$$

For $a \neq 0$ one obtains (2.27) which is equivalent to (2.6). For $a = 0$ the transformation $t = -u',\ u = -t' - x',\ x = x',\ y = u' + y'$ put K into the form (2.5).

If exactly two of the constants a_1, a_2, b_1, b_2 vanish we can obtain either

$$a_1 = a_2 = 0, \qquad b_1 b_2 \neq 0 \tag{2.30}$$

or

$$b_1 = a_2 = 0, \qquad a_1 b_2 \neq 0 \tag{2.31}$$

by an interchange of coordinates. In the case (2.30) the transformation $t' = b_2 t + b_4$, $u' = b_1 u + b_3,\ x' = b_2 x,\ y' = b_1 y$ leads to

$$K = (x + u)(c_1 \partial_u + c_2 \partial_x) + (t + y)(c_2 \partial_t + c_1 \partial_y), \tag{2.32}$$

where $c_1 = b_1 a_3,\ c_2 = b_2 a_4$. In virtue of the transformation $t' = t - y,\ u' = x + u$, $x' = x - u,\ y' = t + y$ one obtains

$$K = (c_2 + c_1)(u\partial_u + y\partial_y) + (c_2 - c_1)(u\partial_x + y\partial_t). \tag{2.33}$$

For $c_1 \neq \pm c_2$ an appropriate rescaling of t and x yields again (2.27). For $c_1 = c_2$ relation (2.33) coincides with (2.6). For $c_2 = -c_1 = c$ we obtain the expression

$$K = 2c(u\partial_x + y\partial_t), \tag{2.34}$$

which transforms to (2.5) under the transformation

$$t = \frac{t'u'}{y'} - x', \qquad u = \frac{1}{y'}, \qquad x = \frac{2cu'}{y'}, \qquad y = \frac{t'}{2cy'}. \tag{2.35}$$

In the case (2.31) we first make the transformation $t' = b_2 t + b_4$, $u' = a_1 u + a_3$, $x' = a_1(b_2 x + b_3)$, $y' = y$, which leads to

$$K = u(x\partial_u + t\partial_y) + (x + c)(t\partial_t + x\partial_x), \tag{2.36}$$

where $c = b_2 a_4 - b_3 a_1$. The vector K takes the form

$$K = -cu\partial_u + (1 - cy)\partial_y \tag{2.37}$$

upon transformation (2.1). For $c \neq 0$ a shift of y yields (2.6). For $c = 0$ one obtains $K = \partial_y$ which is equivalent to (2.5).

If exactly three of the constants a_1, a_2, b_1, b_2 vanish we can assume that $b_1 \neq 0$, $a_1 = a_2 = b_2 = 0$. The transformation $t' = t$, $u' = b_1 u + b_3$, $x' = x$, $y' = b_1 y + b_4$ leads to K of the form (2.33), hence (2.5) or (2.6) follows.

If $a_1 = a_2 = b_1 = b_2 = 0$ then K corresponds to a null translation and it can be easily transformed to (2.5).

Thus, any field of the form (2.4) can be transformed to (2.5) or (2.6) by means of a conformal transformation. To show that it can be done by a conformal transformation which preserves an orientation it is sufficient to note that ∂_u is invariant under the transformation

$$t' = t, \qquad u' = u, \qquad x' = y, \qquad y' = x \tag{2.38}$$

and that $u\partial_u + y\partial_y$ is invariant, modulo the factor -1, under the transformation

$$t' = s^{-2}u, \qquad u' = s^{-2}t, \qquad x' = s^{-2}y, \qquad y' = s^{-2}x \tag{2.39}$$

where $s^2 = tu - xy$. Transformation (2.38) and (2.39) change an orientation of $R^{2,2}$. Given a conformal transformation which transforms K into (2.5) or (2.6) one can always combine it with (2.38) or (2.39) in order to obtain a transformation which preserves the orientation.

The fields ∂_u and $a(u\partial_u + y\partial_y)$ are not related by any conformal transformation. This can be proved by solving the condition $\partial_{u'} = a(u\partial_u + y\partial_y)$ for a coordinate transformation and showing that such transformation cannot be conformal. □

Given the classification of null infinitesimal conformal isometries of $R^{2,2}$, one can use it to find two-dimensional algebras of vector fields (2.4). If these vectors are orthogonal to each other we will call the corresponding algebra a totally null two-dimensional subalgebra of the conformal algebra.

Theorem 2. *Every totally null two-dimensional subalgebra of the conformal algebra of $R^{2,2}$ is equivalent, modulo a conformal transformation which preserves an orientation, to one of the following*

$$\mathrm{Span}\{\partial_u, \partial_y\}, \qquad \mathrm{Span}\{\partial_u, x\partial_u + t\partial_u\}, \qquad \mathrm{Span}\{\partial_u, u\partial_u + y\partial_y\}, \tag{2.40}$$

$$\mathrm{Span}\{\partial_u, \partial_x\}, \qquad \mathrm{Span}\{\partial_u, y\partial_u + t\partial_x\}, \qquad \mathrm{Span}\{\partial_u, u\partial_u + x\partial_x\}, \tag{2.41}$$

Tangent planes defined by (2.40) and (2.41) are, respectively, self-dual and antiself-dual.

Proof. One can assume the following commutation relations for a basis (K_1, K_2) of a subalgebra under consideration

$$[K_1, K_2] = \epsilon K_1, \qquad \epsilon = 0, 1. \tag{2.42}$$

Due to theorem 1 there are coordinates x^μ such that K_1 is given by (2.5) or (2.6). In these coordinates K_2 takes the general form (2.4). If K_1, K_2 are to define self-dual totally null planes K_2 has to take the form

$$K_2 = (b_1 u + b_2 x + b_3)\partial_u + (b_1 y + b_2 t + b_4)\partial_y. \tag{2.43}$$

For K_2 given by (2.43) and K_1 given by (2.6) equation (2.42) cannot be satisfied. For $K_1 = \partial_u$ equation (2.42) yields $b_1 = \epsilon$. For $\epsilon = b_2 = 0$ one obtains the algebra Span$\{\partial_u, \partial_y\}$. For $\epsilon = 0, b_2 \neq 0$ a simple shift in x and t leads to the algebra Span$\{\partial_u, x\partial_u + t\partial_y\}$. For $\epsilon = 1$ one can easily get rid of b_3 and b_4. If $b_2 = 0$ then the algebra Span$\{\partial_u, u\partial_u + y\partial_y\}$ follows. For $b_2 \neq 0$ one obtains the same algebra due to the transformation $t' = t$, $u' = u + b_2 x$, $x' = x$, $y' = y + b_2 t$. The first two algebras in (2.4) are Abelian and the third is non-Abelian. The Abelian ones are nonequivalent since there is no conformal transformation of coordinates such that $\partial_u, x\partial_u + t\partial_y$ are spanned by $\partial_{u'}$ and $\partial_{y'}$.

The algebras (2.41) are obtained from (2.40) by the transformation $t' = t, u' = u, x' = y, y' = x$ which changes an orientation (hence self-dual planes become antiself-dual). □

3. Reductions of the self-duality equations

In terms of the gauge potentials $A_\mu = (A_t, A_u, A_x, A_y)$ the self-duality equations (1.4) take the form

$$\partial_u A_x - \partial_x A_u + [A_u, A_x] = 0, \tag{3.1}$$

$$\partial_t A_y - \partial_y A_t + [A_t, A_y] = 0, \tag{3.2}$$

$$\partial_t A_u - \partial_u A_t + [A_t, A_u] + \partial_x A_y - \partial_y A_x + [A_x, A_y] = 0. \tag{3.3}$$

In this section we consider reductions of equations (3.1)–(3.3) for the Yang–Mills fields invariant under one- or two-dimensional algebras given in section 2 (see (2.5), (2.6), (2.40), (2.41)). For all the algebras except (2.40) the reduced equations are related to a linear system which follows from (1.5), (1.6) under the assumption that ψ is annihilated by vector fields from the algebra (this is also true for all other one- or two-dimensional subalgebras of the conformed algebra).

We say that the Yang–Mills field A is invariant with respect to an algebra generated by vector fields K_i iff

$$L_{K_i} A = 0 \tag{3.4}$$

in some gauge. (Note that for the considered algebras condition (3.4) is equivalent to the more general requirement that A is preserved by K_i up to gauge transformations [HSV].)

If $K = \partial_u$ then it follows from (3.4) that all the functions A_μ are independent of u. Due to (1.2) one can impose the gauge condition

$$A_x = 0 \tag{3.5}$$

(an alternative gauge condition is considered in [MW, section 5.3]). Then equation (3.1) yields

$$A_u = B(t, y). \tag{3.6}$$

It follows from (3.2) that there is a function $J(t,x,y) \in \boldsymbol{G}$ such that

$$A_t = J^{-1}\partial_t J, \qquad A_y = J^{-1}\partial_y J. \tag{3.7}$$

Equation (3.3) yields the following condition for J

$$\partial_x(J^{-1}\partial_y J) + [J^{-1}\partial_t J, B] + \partial_t B = 0. \tag{3.8}$$

Thus, for the symmetry $K = \partial_u$ equations (3.1)–(3.3) reduce to (3.8), where $B(t,y)$ can be chosen arbitrarily.

For $K = \partial_u$ and $G = SL(2,C)$ a more sophisticated reduction of the self-duality equations can be obtained as follows [W3,S] (see also section 5.3 in [MW]). In this case the freedom of gauge transformations (1.2) with $g = g(t,y)$ allows us to transform A_u into one of the following forms

$$A_u = 0, \tag{3.9}$$

$$A_u = \mathrm{i}f\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \qquad f = f(t,y), \tag{3.10}$$

$$A_u = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}. \tag{3.11}$$

In the case (3.9) equations (3.1)–(3.3) reduce to linear ones. In the case (3.10) one obtains the following generalization [Z] of the NLS equation

$$\mathrm{i}\phi_{,t} + \tfrac{1}{2}\phi_{,xy} + \rho\phi = 0, \tag{3.12}$$

$$-\mathrm{i}\tilde{\phi}_{,t} + \tfrac{1}{2}\tilde{\phi}_{,xy} + \rho\tilde{\phi} = 0, \tag{3.13}$$

where

$$\rho_{,y} = (\phi\tilde{\phi})_{,x}. \tag{3.14}$$

In the case (3.11) equations (3.1)–(3.3) reduce either to a system of ordinary and linear equations or to the following generalization of the KdV equation

$$4\phi_{,ty} + \phi_{,xyyy} - 8\phi_{,y}\phi_{,xy} - 4\phi_{,x}\phi_{,yy} = 0 \tag{3.15}$$

(the corresponding equation (5.3.5) in [MW] contains an error).

For $K = u\partial_u + y\partial_y$ the symmetry condition (3.4) yields

$$A_t = \tilde{A}_t, \qquad A_u = (uy)^{-1/2}\tilde{A}_u, \qquad A_x = \tilde{A}_x, \qquad A_y = (uy)^{-1/2}\tilde{A}_y \tag{3.16}$$

where $\tilde{A}_\mu = \tilde{A}_\mu(t,x,z)$ and $z = y/u$. By means of a gauge transformation with $g = g(t,x,z)$ one obtains (3.5). Then, from (3.1) it follows that $\partial_x A_u = 0$. Using again (1.2) (now with some $g = g(t,z)$) leads to

$$A_x = A_u = 0. \tag{3.17}$$

In virtue of (3.16) and (3.17) equations (3.2) and (3.3) yield

$$\partial_t\tilde{A}_y - z^{1/2}\partial_z\tilde{A}_t + [\tilde{A}_t, \tilde{A}_y] = 0, \tag{3.18}$$

$$z^{3/2}\partial_z\tilde{A}_t + \partial_x\tilde{A}_y = 0. \tag{3.19}$$

It follows from (3.18) that there is a function $J(t,x,z)$ such that

$$A_t = J^{-1}\partial_t J, \qquad A_y = u^{-1}J^{-1}\partial_z J. \tag{3.20}$$

In terms of J, equation (3.19) takes the following form

$$(\partial_x + z\partial_t)(J^{-1}\partial_z J) + z[J^{-1}\partial_t J, J^{-1}\partial_z J] = 0. \tag{3.21}$$

Thus, for the symmetry (2.6), the self-duality equations (3.1)–(3.3) reduce to equation (3.21) for a function $J(t, x, z) \in \boldsymbol{G}$. It seems that in this case one cannot obtain equations similar to (3.12)–(3.15) (for $G = SL(2, C)$).

The invariance of A with respect to one of the algebras (2.40) yields, respectively,

$$A_\mu = \tilde{A}_\mu, \tag{3.22}$$

$$A_t = \tilde{A}_t - yt^{-2}\tilde{A}_y, \qquad A_u = t^{-1}\tilde{A}_u, \qquad A_x = \tilde{A}_x - yt^{-2}\tilde{A}_u, \qquad A_y = t^{-1}\tilde{A}_y, \tag{3.23}$$

$$A_t = \tilde{A}_t, \qquad A_u = y^{-1}\tilde{A}_u, \qquad A_x = \tilde{A}_x, \qquad A_y = y^{-1}\tilde{A}_y, \tag{3.24}$$

where $\tilde{A}_\mu = \tilde{A}_\mu(t, x)$. In all the cases the self-duality equations (3.1)–(3.3) reduce to the following system (equivalent to that in section 6.5 in [MW])

$$\partial_x\tilde{A}_u + [\tilde{A}_x, \tilde{A}_u] = 0, \tag{3.25}$$

$$\partial_t\tilde{A}_y + [\tilde{A}_t, \tilde{A}_y] = 0, \tag{3.26}$$

$$\partial_t\tilde{A}_u + [\tilde{A}_t, \tilde{A}_u] + \partial_x\tilde{A}_y + [\tilde{A}_x, \tilde{A}_y] = 0. \tag{3.27}$$

In order to simplify equations (3.25)–(3.27) we assume the gauge condition

$$\tilde{A}_x = 0 \tag{3.28}$$

(note that relations (3.22)–(3.24) are preserved by transformation (1.2) with $g = g(t, x)$). In this gauge equation (3.25) yields

$$\tilde{A}_u = B(t), \tag{3.29}$$

where B is a Lie algebra-valued function of t. Equation (3.26) can be solved by introducing a potential $J(t, x)$ such that

$$\tilde{A}_t = J^{-1}\partial_t J, \qquad \tilde{A}_y = J^{-1}C(x)J, \tag{3.30}$$

where C is a Lie algebra valued function of x. Equation (3.27) yields

$$[J^{-1}\partial_t J, B] + \partial_x(J^{-1}CJ) + \partial_t B = 0. \tag{3.31}$$

Thus, in the case of symmetries (2.4), the self-duality equations (3.1)–(3.3) reduce to equation (3.31) for $J(t, x)$, $B(t)$ and $C(x)$. For these algebras one cannot reduce the linear equations (1.5), (1.6) by assuming $K_1(\psi) = K_2(\psi) = 0$ since then equation (1.5) would imply the constraints $A_u = A_y = 0$ which do not follow from (3.1)–(3.3). Equation (3.31) is related to a pair of Lax operators in a similar way as in the case of completely integrable ordinary equations.

For the gauge group $SU(2)$ equations (3.25)–(3.27) can be integrated explicitly as follows. As before, we assume the gauge condition (3.28) and we obtain (3.29). Equations (3.26) and (3.27) can be considered as algebraic linear equations for $\tilde{A}_t$, which are solvable iff the following equations are satisfied

$$\mathrm{Tr}(\tilde{A}_y\partial_t\tilde{A}_y) = 0, \tag{3.32}$$

$$\mathrm{Tr}(\tilde{A}_u\partial_t\tilde{A}_u + \tilde{A}_u\partial_x\tilde{A}_y) = 0, \tag{3.33}$$

$$\mathrm{Tr}(\tilde{A}_u\partial_t\tilde{A}_y) + \mathrm{Tr}(\tilde{A}_y\partial_t\tilde{A}_u + \tilde{A}_y\partial_x\tilde{A}_y) = 0. \tag{3.34}$$

Equations (3.32)–(3.34) are equivalent to the following algebraic conditions for B and $\tilde{A}_y$

$$\mathrm{Tr}(B)^2 = c_1t^2 + 2c_2t + c_3, \tag{3.35}$$

$$\mathrm{Tr}(B\tilde{A}_y) = -c_1tx - c_2x + c_4t + c_5, \tag{3.36}$$

$$\mathrm{Tr}(\tilde{A}_y)^2 = c_1x^2 - 2c_4x + c_6, \tag{3.37}$$

where c_i are real constants. Thus, the self-duality equations for $SU(2)$ gauge fields with symmetries (2.40) do not reduce to interesting differential equations. We do not know whether this is also true for other gauge groups however our results (unpublished) for $SL(2, C)$ and $SU(3)$ are also discouraging (we expect similar results for any gauge algebra which admits an ad-invariant scalar product).

The invariance of the Yang–Mills field A with respect to one of the algebras (2.41) yields, respectively,

$$A_\mu = \tilde{A}_\mu(t, y), \tag{3.38}$$

$$A_t = \tilde{A}_t - x\tilde{A}_x, \qquad A_u = t\tilde{A}_u, \qquad A_x = t\tilde{A}_x, \qquad A_y = \tilde{A}_y - x\tilde{A}_u, \tag{3.39}$$

$$A_t = \tilde{A}_t, \qquad A_u = x^{-1}\tilde{A}_u, \qquad A_x = x^{-1}\tilde{A}_x, \qquad A_y = \tilde{A}_y, \tag{3.40}$$

where $\tilde{A}_\mu = \tilde{A}_\mu(t, y)$. Equations (3.1)–(3.3) take the following form

$$[\tilde{A}_x, \tilde{A}_u] = \epsilon\tilde{A}_u, \tag{3.41}$$

$$\partial_t\tilde{A}_y - \partial_y\tilde{A}_t + [\tilde{A}_t, \tilde{A}_y] = 0, \tag{3.42}$$

$$\partial_t\tilde{A}_u + [\tilde{A}_t, \tilde{A}_u] - \partial_y\tilde{A}_x - [\tilde{A}_y, \tilde{A}_x] = 0, \tag{3.43}$$

where $\epsilon = 0$ in the cases (3.28) and (3.39) and $\epsilon = 1$ in the case (3.40). It follows from (3.42) that

$$\tilde{A}_t = \tilde{A}_y = 0 \tag{3.44}$$

in some gauge. From (3.44) and (3.43) one obtains

$$\tilde{A}_u = \partial_y R, \qquad \tilde{A}_x = \partial_t R, \tag{3.45}$$

where $R = R(t, y)$ is a Lie algebra-valued function. It has to satisfy the equation [T]

$$[\partial_t R, \partial_y R] = \epsilon\partial_y R, \tag{3.46}$$

which follows from (3.41). Thus, for symmetries (2.41), the self-duality equations (3.1)–(3.3) reduce to equation (3.46) for $R(t, y)$, where $\epsilon = 0, 1$.

For $\epsilon = 1$ and a compact gauge group equation (3.46) admits only the trivial solutions $R = R(y)$. For $\epsilon = 0$ and $G = SU(2)$ equation (3.46) is also trivial since it is equivalent to $R = R(y)$ or to $\partial_y R + f\partial_t R = 0$, where f is a real function. However, in other cases it might be interesting. For instance, when $\epsilon = 0$, $G = SU(3)$ and $\partial_y R$ is generic in the sense that it has three distinct eigenvalues, equation (3.46) is equivalent to the nonlinear equation

$$\partial_t R = f_1\partial_r R + \mathrm{i} f_2((\partial_y R)^2 - \tfrac{1}{3}\,\mathrm{Tr}(\partial_y R)^2), \tag{3.47}$$

where f_1, f_2 are real functions which can be arbitrarily prescribed. Equation (3.47) can be replaced by the following quasilinear equation for $Q = \partial_y R$

$$\partial_t Q = \partial_y(f_1 Q + \mathrm{i} f_2(Q^2 - \tfrac{1}{3}\,\mathrm{Tr}\, Q^2)). \tag{3.48}$$

Equations (3.41)–(3.43) become more interesting when supplemented by algebraic conditions on gauge fields. In particular, in this way one can obtain (for $\epsilon = 0$) the Boussinesq equation and the n-wave equation (see section 6.4 in [MW] and references therein). One can try to apply analogous techniques when $\epsilon = 1$.

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