Dimension of interaction dynamics

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A method allowing one to distinguish interacting from noninteracting systems based on available time series is proposed and investigated. Some facts concerning generalized Rényi dimensions that form the basis of our method are proved. We show that one can find the dimension of the part of the attractor of the system connected with interaction between its parts. We use our method to distinguish interacting from noninteracting systems on the examples of logistic and Hénon maps. A classification of all possible interaction schemes is given.

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I. INTRODUCTION

Given two time series can one tell if they originated from interacting or noninteracting systems? We show that with the help of embedding methods [1], Takens' theorem [2,3], and some facts concerning Rényi dimensions that we prove, one can succeed in the case of chaotic systems. Moreover, one can quantify the common part of the dynamics, which we call the dynamics of interaction.

It happens sometimes, especially in simple systems like electronic circuits or coupled mechanical oscillators, that one knows whether the systems under investigation are coupled or not, and the direction and sometimes strength of the coupling. However, there are many complex phenomena in nature where one is unable to verify directly the existence of coupling between parts of the system in which the phenomenon takes place. Especially in complex spatiotemporal systems, like fluid systems, brain, neuronal tissue, social systems, etc., one often faces the problems of characterization of the interdependence of parts of the system of interest and of quantifying the strength of interactions between the parts.

Recent research in neurology, for example, has shown that temporal coordination between different, often distant neural assemblies plays a critical role in the neurophysiological underpinnings of such cognitive phenomena as the integration of features in object representation (cf. [4] for a review) and the conscious experience of stimuli [5]. The critical empirical question, therefore, is which of the neural assemblies synchronize their activity. Since coordination may take many forms, including complex nonlinear relations, simple correlational methods may not be sufficient to detect it. The detection of nonlinear forms of coordination is also critically important for issues in cognitive science [6], developmental psychology [7], and social psychology [8,9].

The method traditionally used for this purpose is correlation analysis. Given two time series one studies their autocorrelation functions and cross correlations. Large cross correlations are usually attributed to large interdependence between the parts. Small cross correlations are considered as the signature of independence of the variables.

Unfortunately, linear time series analysis gives meaningful results only in the case of linear systems or stochastic time series. It is well known that spectral analysis alone cannot discriminate between low-dimensional nonlinear deterministic systems and stochastic systems [10], even though the properties of the two kinds of system are different.

Methods based on entropy measures represent one viable approach for detecting nonlinear relations between the activity of different neural assemblies [5]. Recently another approach based on nonlinear mutual prediction has been proposed and used in an experiment. Pecora, Carroll, and Heagy [11] developed a statistics to study the topological nature of functional relationships between coupled systems. Schiff *et al.* [12] used it as a basis of their method. The idea is as follows. If there exists a functional relationship between two systems, it is possible to predict the state of one system from the known states of the other. This happens if the coupling between two systems is strong enough so that generalized synchronization occurs [11,13,14]. The average normalized mutual prediction error is used to quantify the strength and directionality of the coupling [12].

The method we introduce in the present paper does not assume generalized synchrony. We introduce the notion of the *dimension of interaction*, which measures the size of the dynamics responsible for the coupling between the two systems. More precisely, it is the dimension of the part of the attractor of the whole system that is acted on by the dynamics of both subsystems. We also show how to obtain information concerning the strength and directionality of the coupling.

The idea is, in fact, very simple. Given two time series from subsystems of interest we construct another one that probes the whole system, for instance, by adding the two series. If the subsystems do not interact, the dimension of the whole system is the sum of the dimensions of the two subsystems, all of which can be estimated from data. On the other hand, if the subsystems have some common degrees of freedom, the dimension of the whole system will be smaller than the sum of the dimensions of the two subsystems.

Our method can also be used to find if two response systems have a common driver. We discuss this application in Sec. III.

The structure of the paper is as follows. In Sec. II we recall the definition of the Rényi dimensions and formulate three theorems that form the basis of our method. The rather straightforward proofs have been relegated to Appendix A, since they are not crucial for understanding the method itself and can be omitted by readers whose main interest is in applications. We formulate our method in Sec. III. Classification of all the possible interaction schemes is given in Sec. IV. A simple way of verifying the kind and direction of the coupling is provided. Results from simulations of coupled logistic and Hénon maps are collected in Sec. V. Final comments and outlook are given in the last section.

II. THEORETICAL CONSIDERATIONS

Our method presented in Sec. III is based on three theorems relating dimensions of subsystems to the dimension of the whole system. The first one states the intuitively obvious fact that the dimension of a system consisting of two noninteracting parts is the sum of the dimensions of the subsystems. The less trivial Theorems 2 and 3 establish interdependencies among the dimensions of the system and its interacting parts. Before we state our theorems we shall recall the definition of the Rényi dimensions.

A. Rényi dimensions

It is at present generally accepted that many objects, both in real physical space and in phase space, are multifractals [15–18]. This means they can be described by (statistically) self-similar probability measures. This usually implies that they can be decomposed into an (infinite) number of objects of different Hausdorff dimensions, or, equivalently, they have nontrivial multifractal spectra of dimensions.

The Rényi dimensions [19] have drawn the attention of physicists and mathematicians since publication of the papers by Grassberger, Hentschel and Procaccia [20–23]. For a probability measure μ on a d-dimensional space U one takes a partition of U into small cells of equal linear size ϵ (equal volume ϵ^d). One defines the Rényi dimensions[1] as

$$D_q(\mu) := \begin{cases} \lim\limits_{\epsilon\to 0} \dfrac{1}{q-1} \dfrac{\ln \Sigma_i p_i^q}{\ln \epsilon} & \text{for} \quad q \in \mathbb{R}\backslash\{1\} \\ \lim\limits_{\epsilon\to 0} \dfrac{\Sigma_i p_i \ln p_i}{\ln \epsilon} & \text{for} \quad q=1 \end{cases}, \tag{1}$$

where

$$p_i = \mu(i\text{th cell}) = \int_{i\text{th cell}} d\mu(x),$$

and the sum is taken over all cells with $p_i \neq 0$.

Of particular importance are D_0, the box-counting dimension, usually equal to the Hausdorff dimension [15,33,34], D_1, the information dimension or the dimension of the measure [19,35,36,24,37], which describes how the entropy $-\Sigma_i p_i \ln p_i$ increases with the change of scale, and D_2, the correlation dimension [22,23,38], which can be most easily extracted from data, and is usually treated as a lower estimate of D_1 since $D_{q_1} \leq D_{q_2}$ for $q_1 > q_2$.

Generalized dimensions are defined for all real q; however, in our proofs we shall restrict our attention to the case $q \geq 1$. We are particularly interested in $q=1$ and 2.

B. Noninteracting systems

Consider two noninteracting dynamical systems $(U_1, \varphi_1, \mu_1), (U_2, \varphi_2, \mu_2)$, where $U_i \subset \mathbb{R}^{n_i}$ is the phase space, φ_i is a flow or a map on U_i, and μ_i is an ergodic φ_i-invariant natural measure on U_i.

Below we shall concentrate on the case of continuous systems. Changes needed for the discrete time case are mostly notational.

By natural measure μ we mean the measure defined as the average over a trajectory

$$\mu = \mu_{x_0} := \lim_{T\to\infty} \frac{1}{T} \int_0^T \delta(x - \varphi_t(x_0)) dt, \tag{2}$$

if it converges (in the weak sense) for μ almost every x_0 and is independent of the trajectory (of the starting point x_0). That is, Eq. (2) is the natural measure if the limit

$$\int f d\mu_{x_0} := \lim_{T\to\infty} \frac{1}{T} \int_0^T \delta(x - \varphi_t(x_0)) f(x) dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_0^T f(\varphi_t(x_0)) dt$$

exists for every continuous function f and is the same for μ almost every x_0 [all x_0 except for the set A such that $\mu(A) = 0$]. In other words, the average of f along a typical trajectory is independent of the trajectory; thus time averages are equal to ensemble averages. One usually thinks of some physical measure, like the Sinai-Ruelle-Bowen measure [39], for which the set of x_0 such that $\mu = \mu_{x_0}$ is of nonzero Lebesgue measure.

The composite noninteracting system has a product structure $(U_1 \times U_2, \varphi_1 \times \varphi_2, \mu_1 \times \mu_2)$. Its dynamics can be written as

$$\mathbf{u}_1(t) = \varphi_1(\mathbf{u}_1(0), t),$$

$$\mathbf{u}_2(t) = \varphi_2(\mathbf{u}_2(0), t).$$

Theorem 1. Suppose $D_q(\mu_1)$, $D_q(\mu_1)$, and $D_q(\mu_1 \times \mu_2)$ exist. Then

[1] An equivalent description of multifractal measures is the f_α spectrum [24–26]. A thorough discussion of the properties of D_q and f_α spectra falls beyond the scope of this paper. Some good reviews of these with discussion of the abundant literature on multifractals can be found, e.g., in [16–18,27–32]. Mathematically precise definitions of multifractal spectra can be found in [17,18].

$$D_q(\mu_1 \times \mu_2) = D_q(\mu_1) + D_q(\mu_2). \tag{3}$$

This means, as should be intuitively obvious, that the dimensions of noninteracting subsystems add up to the dimension of the whole system. The proof is given in Appendix A.

This is in fact one of the long-standing problems in dimension theory, namely, finding the conditions under which the equality holds for various dimensions for arbitrary measures. Some results for Olsen's version of multifractal formalism with a discussion of previous results can be found in [40].

C. Interacting systems

Take two interacting subsystems U_1 and U_2 of system U. It may happen that all the variables in U_1 couple with all those in U_2 but this is not necessary. For many-dimensional systems the structure of the equations of dynamics can be very complicated.

Consider the following decomposition of variables of U_i. Let $\mathbf{y}_1$ be the largest set of variables in U_1 satisfying the condition that if their state is changed whatsoever it will not influence the future evolution of U_2. Similarly define $\mathbf{y}_2$. Put all the remaining variables of U_1, U_2 in vector $\mathbf{x}$. They form a dynamical system V—the part of the whole system that is responsible for the interaction. Then the dynamics of the whole system U can be written as

$$\dot{\mathbf{x}} = f(\mathbf{x}),$$

$$\dot{\mathbf{y}}_1 = g_1(\mathbf{x}, \mathbf{y}_1), \tag{4}$$

$$\dot{\mathbf{y}}_2 = g_2(\mathbf{x}, \mathbf{y}_2).$$

Thus the dynamics of the interacting systems U_1 and U_2 is formally equivalent to the dynamics of three systems: X (interaction part) driving Y_1 and Y_2. We pursue this analogy deeper in the next section. An example where such a decomposition arises naturally is given in Appendix B.

Note that at this point we do not assume anything specific about the dimensionality of the variables x, y_1, and y_2. In particular, it may happen that the set of x's and/or one or both of the sets of variables y_1 or y_2 can be empty. In this case the problem reduces to a simple system in which one cannot specify an interesting subsystem or two noninteracting systems. All these cases are specific instances of the general scheme presented in Sec. IV.

Let $\mu_U, \mu_1, \mu_2, \mu_V$ be natural measures of the dynamical systems U, U_1, U_2, V, respectively (the notation used in this section is gathered in Table I).

Theorem 2. Suppose $D_1(\mu_1)$, $D_1(\mu_2)$, $D_1(\mu_V)$, and $D_1(\mu_U)$ exist. Then

$$D_1(\mu_V) \leqslant d_{\text{int}} := D_1(\mu_1) + D_1(\mu_2) - D_1(\mu_U). \tag{5}$$

(We shall call d_{int} the dimension of interaction.) The equality holds when $\mathbf{y}_1$ and $\mathbf{y}_2$ are asymptotically independent.

Asymptotic independence means essentially a lack of generalized synchronization between the $\mathbf{y}$'s and their common driver $\mathbf{x}$. We relegate further discussion to Appendix A, where we make this condition precise and show where it is needed.[2]

TABLE I. Notation.

System	Variables	Evolution	Measure
V	$(\mathbf{x})$	$\dot{\mathbf{x}} = f(\mathbf{x})$	$\mu_V(\mathbf{x})$
U_1	$(\mathbf{x}, \mathbf{y}_1)$	$\dot{\mathbf{x}} = f(\mathbf{x})$ $\dot{\mathbf{y}}_1 = g_1(\mathbf{x}, \mathbf{y}_1)$	$\mu_1(\mathbf{x}, \mathbf{y}_1)$
U_2	$(\mathbf{x}, \mathbf{y}_2)$	$\dot{\mathbf{x}} = f(\mathbf{x})$ $\dot{\mathbf{y}}_2 = g_2(\mathbf{x}, \mathbf{y}_2)$	$\mu_2(\mathbf{x}, \mathbf{y}_2)$
U	$(\mathbf{x}, \mathbf{y}_1, \mathbf{y}_2)$	$\dot{\mathbf{x}} = f(\mathbf{x})$ $\dot{\mathbf{y}}_1 = g_1(\mathbf{x}, \mathbf{y}_1)$ $\dot{\mathbf{y}}_2 = g_2(\mathbf{x}, \mathbf{y}_2)$	$\mu_U(\mathbf{x}, \mathbf{y}_1, \mathbf{y}_2)$

If we think of dimensions as estimates of the number of degrees of freedom, Theorem 2 means intuitively that, if the system can be considered as composed of interacting parts, some of the degrees of freedom—perhaps even all—are common for both parts. Therefore, the dimension of the whole system is equal to the sum of the number of the common degrees of freedom, those degrees of freedom which belong to U_1 and do not belong to U_2, and the other way around. Thus if we add the dimensions of the subsystems U_1 and U_2, we count the common degrees of freedom twice. We must therefore subtract them if we want to get the dimension of the whole system U.

In the above theorem we show that this intuition can be made precise in the case of the information dimension D_1 and with an additional assumption. The notions of the dimension of interaction that we define in Eq. (5) and of the generalized dimensions of interaction defined below are crucial for our method.

In the special case when one (or both) of $U_i = V$ (all the variables of U_i couple with some of the variables of the other subsystem), say, $U_2 = V$, we may establish the following theorem.

Theorem 3. Suppose $D_q(\mu_1)$, $D_q(\mu_2)$, and $D_q(\mu_V)$ exist and $U_2 = V$. Then

$$D_q(\mu_V) = d_q^{\text{int}} := D_q(\mu_1) + D_q(\mu_2) - D_q(\mu_U). \tag{6}$$

The proof is obvious, for in this case $U_2 \equiv V$ and $U_1 U$. This also means that the above intuitions in this case are precise for arbitrary generalized dimensions and no further assumptions are needed.

The generalized dimensions of interaction d_q^{int} are estimates of the number of effective degrees of freedom responsible for the interaction between the parts of the system un-

[2]One would like to establish a similar inequality in the case of other Rényi dimensions; however, in general, even when $\mathbf{y}_1$ and $\mathbf{y}_2$ are asymptotically independent, $D_q(\mu_V) + D_q(\mu_U) - D_q(\mu_1) - D_q(\mu_2)$ can have arbitrary sign (cf. Appendix A). Nevertheless, we expect this difference for typical physical systems to be small in comparison with the dimensions involved, and typically $D_q(\mu_V) \leqslant d_q^{\text{int}} = D_q(\mu_1) + D_q(\mu_2) - D_q(\mu_U)$.

der study. Of most interest are $d_1^{\rm int}\equiv d_{\rm int}$, which has the best analytical properties, and $d_2^{\rm int}$, which can be most reliably estimated from data.

Note that

$$\begin{aligned}\max\{d_q^{\rm int},D_q(\mu_V)\}&\leqslant\min\{D_q(\mu_1),D_q(\mu_2)\}\\&\leqslant\max\{D_q(\mu_1),D_q(\mu_2)\}\\&\leqslant D_q(\mu_U).\end{aligned}\tag{7}$$

Furthermore, for $q=1$ one can show that

$$0\leqslant D_1(\mu_V)\leqslant d_1^{\rm int}.$$

We conjecture $d_q^{\rm int}\geqslant 0$ also for $q>1$. We also expect typically $d_q^{\rm int}\geqslant D_q(\mu_V)$.

III. METHOD

Suppose we are given two time series measured in subsystems U_1 and U_2 of system U whose structure and interdependence we do not know, e.g., signals gathered on two electrodes placed in not too distant portions of the brain, or measurements of velocity or temperature in various parts of a moderately turbulent fluid. We would like to know if the equations governing the dynamics of both of these variables are coupled or not, how many degrees of freedom are common, and what is the direction of the coupling.

Let X_i be a function on U_i, i.e., $X_i:U_i\to\mathbb{R}$. The time series we measure are $x_1(n):=X_1(\mathbf{u}_1(t_n))$ and $x_2(n):=X_2(\mathbf{u}_2(t_n))$. Let $Y:\mathbb{R}^2\to\mathbb{R}$ be a smooth function nontrivially depending on both variables.[3] Therefore we exclude functions for which $\partial Y/\partial\mathbf{x}_1=0$ or $\partial Y/\partial\mathbf{x}_2=0$. We construct another time series $y(n)=Y(x_1(n),x_2(n))$. Thus $Y(X_1,X_2)$ is a function on U.

Using the time delay method [1,2] we can reconstruct the dynamics of the systems U_i and U from $x_i(n)$ and $y(n)$. That is, for a given *delay* τ and *embedding dimension* N we construct *delay vectors*

$$\tilde{\mathbf{u}}_1(n)=(x_1(n),x_1(n-\tau),\ldots,x_1(n-(N-1)\tau));$$

the construction of $\tilde{\mathbf{u}}_2$ from x_2 and $\tilde{\mathbf{u}}$ from y is similar.

[3]For finite noisy time series some functions are better than others. One cannot promote one function over another. Our experience shows that linear combinations usually provide the most reliable results. There were some cases, however, where other functions were preferred. This is natural if one realizes that the variables measured may have completely different physical meaning, such as temperature and pressure, say. In this case a sum is not the most natural combination of the two time series. In practice we used five different functions $Y(x,y)$, namely, $x+y$, xy, $\sin(x)\cos(y)$, $x\exp(y)$, and $2x-y$, to calculate the dimension of the system $D_q(\mu_U)$, and averaged the results. The variance of the five estimates obtained was usually small. These functions were not chosen for their particularly good numerical properties but rather to verify that the results obtained depend only weakly on the choice of the function Y.

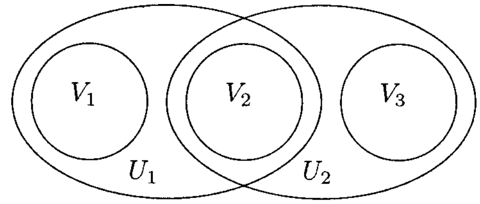

FIG. 1. Simple interaction.

If $N>2D_0(\mu_1)$, for all reasonable delays, for infinite not-too-sparsely-probed time series, the Takens theorem [2,3] guarantees that $\tilde{\mathbf{u}}_1(n)$ is an embedding of the original invariant set in U_1. To calculate dimensions it is even enough to take $N>D_0(\mu_1)$ [41,42]. It is generally believed that for finite but not too short and not too noisy time series also the above construction occasionally gives a reasonable estimate of the original dynamics. For a detailed discussion of these issues the reader should consult the relevant literature, e.g., [10,43–46]. We disregard the practical problems until Sec. V, where we show some numerical results. For the time being we discuss clean infinite time series.

Having reconstructed the attractors we can estimate their generalized dimensions and calculate the generalized dimensions of interaction

$$d_q^{\rm int}:=D_q(\mu_1)+D_q(\mu_2)-D_q(\mu_U).\tag{8}$$

It is also convenient to consider normalized dimensions of interaction:

$$\begin{aligned}m_1^q&:=d_q^{\rm int}/D_q(\mu_1),\\m_2^q&:=d_q^{\rm int}/D_q(\mu_2),\\m_U^q&:=d_q^{\rm int}/D_q(\mu_U).\end{aligned}\tag{9}$$

From the values of m_i^q we can infer the information we need. All the possible cases are described in the next section. Note that if $m_i^q\neq 0$ they satisfy

$$\frac{1}{m_1^q}+\frac{1}{m_2^q}-\frac{1}{m_U^q}=1.$$

From Eq. (7) we also have

$$0\leqslant m_U^q\leqslant m_1^q,m_2^q\leqslant 1,$$

which provides us with a tool to check the consistency of results.

Before we present the classification of all the possible schemes of interaction let us discuss heuristically four simple examples.

(i) If U_1 and U_2 are uncoupled, the variables we see through x_1 and x_2 are different; thus $\mu_U=\mu_1\times\mu_2$. Therefore, from Theorem 1, $d_q^{\rm int}=0$, as it should be for any reasonable definition of dimension of interaction for noninteracting systems.

(ii) Consider now a system U consisting of three isolated systems V_i (see Fig. 1), which we cannot observe separately, but rather through U_1 and U_2, e.g., by measuring $X_1(\mathbf{v}_1,\mathbf{v}_2)$

and $X_2(\mathbf{v}_2,\mathbf{v}_3)$. Reconstructing the dynamics from time series of X_1 and X_2 we expect to obtain

$$D_q(\mu_1)=D_q(\mu_{V_1})+D_q(\mu_{V_2}),$$

$$D_q(\mu_2)=D_q(\mu_{V_2})+D_q(\mu_{V_3}).$$

With a typical function $Y(x_1,x_2)$ we obtain a time series $y(n)$ from which we estimate

$$D_q(\mu_U)=D_q(\mu_{V_1})+D_q(\mu_{V_2})+D_q(\mu_{V_3}).$$

Since the dynamics of V_2 is responsible for the interaction between U_1 and U_2, we want to call the dimension of μ_{V_2} the dimension of interaction. According to the definition (8) we have

$$\begin{aligned} d_q^{\text{int}}&=D_q(\mu_1)+D_q(\mu_2)-D_q(\mu_U)\\ &=D_q(\mu_{V_1})+D_q(\mu_{V_2})+D_q(\mu_{V_2})+D_q(\mu_{V_3})\\ &\quad-[D_q(\mu_{V_1})+D_q(\mu_{V_2})+D_q(\mu_{V_3})]\\ &=D_q(\mu_{V_2}). \end{aligned}$$

(iii) Consider now the general situation described in Sec. II C. Reconstructing the dynamics from time series of typical variables from systems U_1 and U_2, say, $x_1(n)$ and $x_2(n)$, we get

$$D_q(\mu_1)\geqslant D_q(\mu_V),$$

$$D_q(\mu_2)\geqslant D_q(\mu_V).$$

From a typical function $Y(x_1,x_2)$ we reconstruct the attractor of U and obtain

$$\begin{aligned} &\max\{D_q(\mu_1),D_q(\mu_2)\}\\ &\quad\leqslant D_q(\mu_U)\\ &\quad\leqslant D_q(\mu_V)+[D_q(\mu_1)-D_q(\mu_V)]\\ &\qquad+[D_q(\mu_2)-D_q(\mu_V)], \end{aligned}$$

where $D_q(\mu_1)-D_q(\mu_V)$ quantifies the number of degrees of freedom in U_1 not coupled to U_2. From this we conclude that

$$\begin{aligned} 0<D_q(\mu_V)&\leqslant d_q^{\text{int}}\\ &=D_q(\mu_1)+D_q(\mu_2)-D_q(\mu_U)\\ &\leqslant\min\{D_q(\mu_1),D_q(\mu_2)\}, \end{aligned}$$

the difference between $D_q(\mu_V)$ and d_q^{int} depending on the strength of synchronization between U_1 and U_2.

(iv) As the last example we shall take a system X driving two response systems Y_1 and Y_2. Suppose we also have a second copy of this setup, namely, we drive X' with response systems Y_1' and Y_2'. We collect simultaneously four time series of some variable from all the response systems. Now we choose two of them randomly and want to know if the systems they come from have a common driver. It is easy to check that, if they have, then d_q^{int} is approximately the dimension of the invariant measure of the driver system $D_q(\mu_X)>0$. If they have different drivers, then $d_q^{\text{int}}=0$.

Summarizing, from measurements involving parts of the given system and arbitrary nontrivial smooth functions of two variables we can reconstruct the dimensions of measures μ_1, μ_2, and μ_U. From this we can obtain the dimension of interaction d_q^{int} [Eq. (8)]. Depending on the values of $D_q(\mu_1)$, $D_q(\mu_2)$, $D_q(\mu_U)$, and d_q^{int} we can determine if the systems are coupled or not, and what is the direction of coupling.

IV. CLASSIFICATION OF POSSIBLE INTERACTION SCHEMES

Let us thus assume that we have two subsystems and their reconstructed dimensions are $D_q(\mu_1)$ and $D_q(\mu_2)$. The dimension of the whole system $D_q(\mu_U)$ is obtained from time series $y(n)$ constructed through the procedure described in the previous section. The dimension of interaction is calculated from Eq. (8). The above discussion leads to the question of what situations are possible. There are four nonequivalent cases, which are conveniently described by the following proposition.

Proposition 4.

(1) If $d_q^{\text{int}}=0$, then $\mu_U=\mu_1\times\mu_2$ (the systems U_1 and U_2 do not interact);

(2) If $D_q(\mu_1)=D_q(\mu_2)=d_q^{\text{int}}$, then $\mu_U=\mu_1\equiv\mu_2\equiv\mu_V$ (the systems U_1 and U_2 are the same system or we have maximal coupling);

(3) If $D_q(\mu_1)>D_q(\mu_2)=d_q^{\text{int}}$, then $\mu_2=\mu_V$ and $\mu_1\equiv\mu_U$ (all variables of U_2 couple to some of the degrees of freedom of U_1, or U_2 is the driver in the pair, which gives $U_1\equiv U$);

(4) In all other cases $D_q(\mu_1),D_q(\mu_2)>d_q^{\text{int}}$, which means interaction or double control (two response systems driven by a common driver).

Note that this proposition is to some extent opposite to the theorems proposed in Sec. II. It can be shown for $q=1$ [47]. We verify it numerically for particular systems for $q=2$ in the next section.

It is convenient sometimes to use m_1^q,m_2^q,m_U^q [Eq. (9)]. We can write the above classification in this case as follows.

(1) $m_1^q=m_2^q=m_U^q=0$ (no interaction);

(2) $1=m_1^q=m_2^q=m_U^q$ (maximal coupling: $\mu_1\equiv\mu_2\equiv\mu_V$);

(3) $1=m_1^q>m_2^q=m_U^q>0$ (all the degrees of freedom of $U_2\equiv V$ couple to some variables of U_1);

(4) $1>m_1^q\geqslant m_2^q>m_U^q>0$ (interaction or double control).

All four cases are presented symbolically in Fig. 2.

The examples considered in the previous section can easily be identified as particular cases of this classification. Thus example (i) represents case 1, example (ii) represents case 4, example (iii) can represent cases 2, 3, or 4, and the last example represents case 1 or 4 depending on whether the signals analyzed come from systems coupled to the same driver or not.

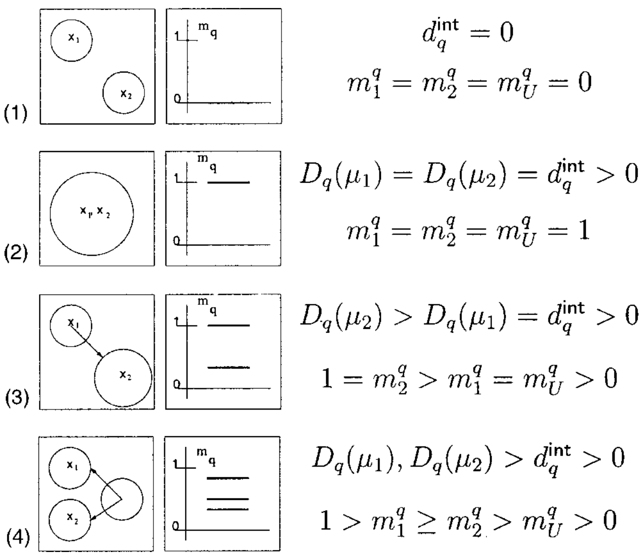

FIG. 2. Classification of possible interaction schemes. The first column shows symbolically the relative position in abstract space of the subsystems in which we measure the time series x_1 and x_2. An arrow from one system to another means the future states of the second system depend on the current states of both. The second column shows the values of normalized dimensions m_1^q, m_2^q, and m_U^q in each case. The numbers refer to the cases in Proposition 4.

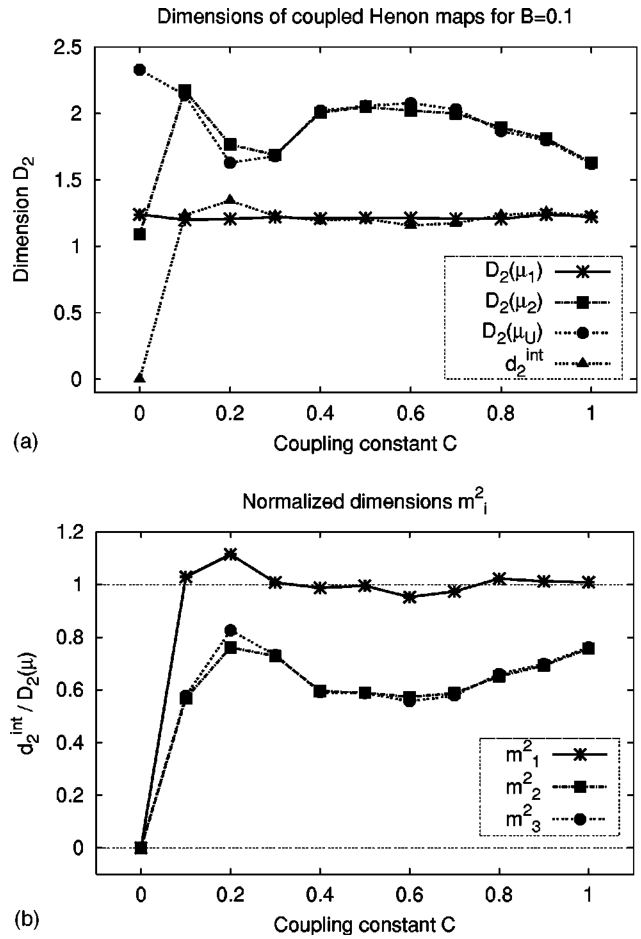

FIG. 3. (a) Dimensions $D_2(\mu_1)$, $D_2(\mu_2)$, $D_2(\mu_U)$, and d_2^{int} of one-way coupled nonidentical Hénon maps [Eq. (10)] for $B=0.1$. (b) Normalized dimensions m_1^2, m_2^2, and m_U^2 for the same systems.

V. NUMERICAL RESULTS

Below we shall present some applications of our method to analysis of numerical results for several paradigmatic systems (coupled Hénon maps and logistic maps).

Throughout this section we will use d_2^{int}. The dimensions presented in the pictures are always D_2 calculated with the help of the d2 program from the TISEAN package [48] with an algorithm that is an extension of algorithms published previously [22,23,38] and improves the speed of computation [48]. In every case we used 10^5 points with one exception described in the text. The functions Y used to calculate the dimension of the whole system (cf. previous sections) were $x+y$, xy, $\sin(x)\cos(y)$, $x\exp(y)$, and $2x-y$. To estimate the dimension we smoothed output from d2 with the help of the Takens-Theiler estimator [10,48–50] c2t and averages over local dimension c2d [10,48].

A. Two Hénon maps

Consider a system U consisting of two Hénon maps [51] coupled as follows [12]:

$$K\begin{cases} x_{i+1}=1.4-x_i^2+0.3y_i, \\ y_{i+1}=x_i, \end{cases} \tag{10}$$

$$L\begin{cases} u_{i+1}+1.4-[Cx_i+(1-C)u_i]u_i+Bv_i, \\ v_{i+1}=u_i. \end{cases}$$

Thus Hénon system K drives system L. The coupling is introduced through the variable u. We consider the case of coupled identical systems ($B=0.3$) and nonidentical coupled systems ($B=0.1$). The parameter C measures the strength of interaction.

Suppose the variables accessible experimentally are x_n and u_n. What can be said in this case about the interaction between systems K and L?

Certainly, for $C=0$ the systems K and L do not interact (case 1 in our classification); therefore $D_q(\mu_U)=D_q(\mu_K)+D_q(\mu_L)$ and $d_q^{\text{int}}=0$. On the other hand, for positive C the influence of x should be reflected in the behavior of u. From Theorem 3 we expect $D_q^{\text{int}}=D_q(\mu_K)$ (case 3). One can also expect that for C slightly above 0, $D_q(\mu_U)$ will not change much, while $D_q(\mu_L)$ should jump from its value at 0 to the value of $D_q(\mu_U)$ at $C=0$.

This behavior can indeed be seen in Fig. 3(a) for nonidentical Hénon systems ($B=0.1$) and in Fig. 4(a) for identical systems ($B=0.3$). The synchronization of x and u [52,53] visible for $C\geqslant 0.7$ (case 2) can be discovered much more simply: if one plots several consecutive values of x_n-u_n versus coupling, for these particular values all the points fall on 0 (Fig. 5; see also Fig. 7 of [12]). Looking at the normalized dimensions [Figs. 3(b) and 4(b)] we easily identify lack

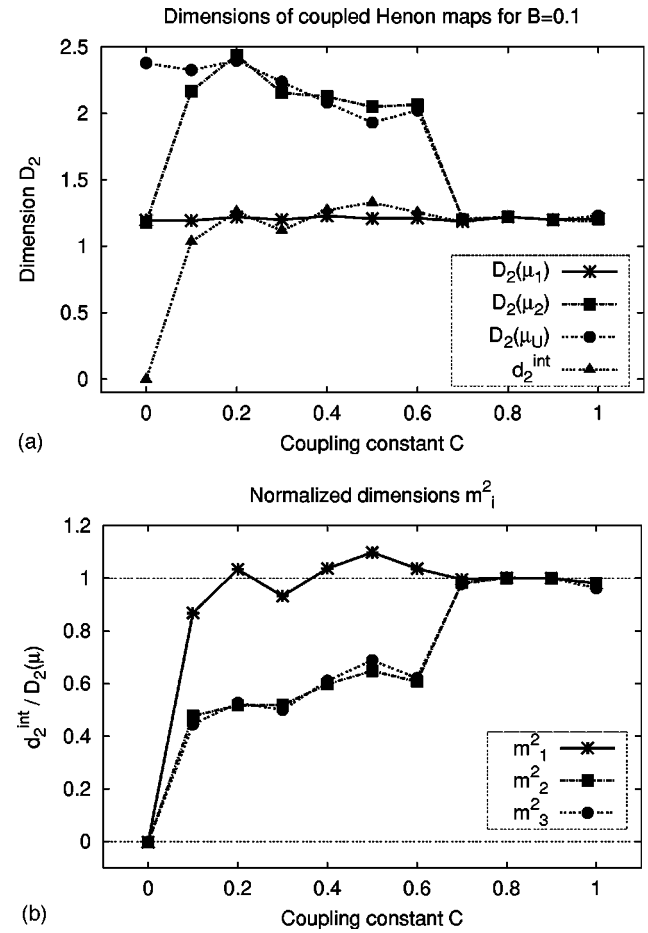

FIG. 4. (a) Dimensions $D_2(\mu_1)$, $D_2(\mu_2)$, $D_2(\mu_U)$, and d_2^{int} of one-way coupled identical Hénon maps [Eq. (10)] for $B=0.3$. (b) Normalized dimensions m_1^2, m_2^2, and m_U^2 for the same systems.

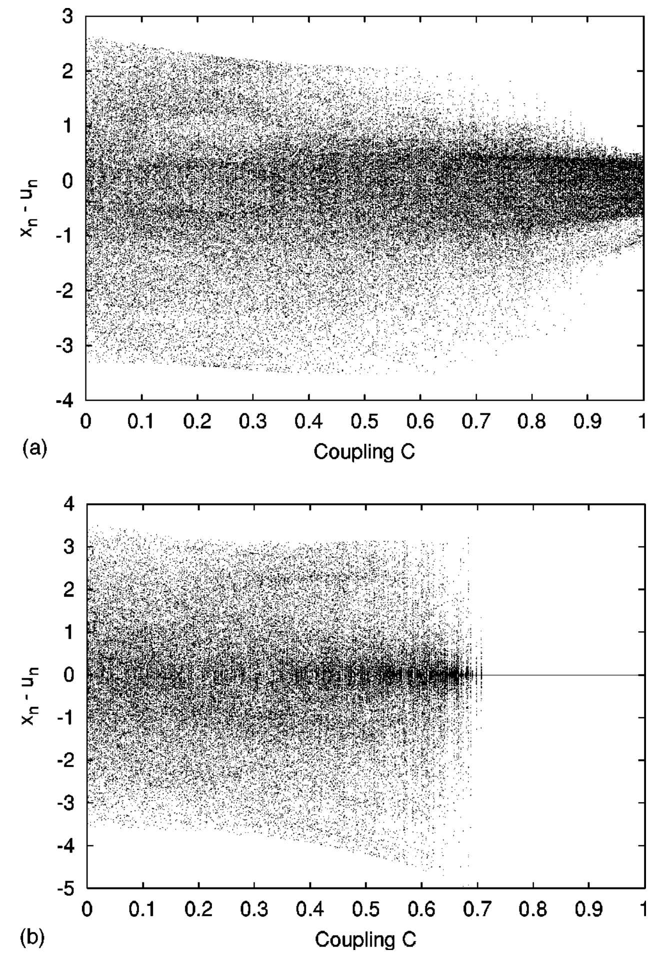

FIG. 5. Differences between the x_n and u_n values of two coupled Hénon systems [Eq. (10)] for 100 consecutive values (a) for the case of nonidentical systems ($B=0.1$) and (b) for the case of identical systems ($B=0.3$).

of coupling for $C=0$ $(m_1=m_2=m_3=0)$, case 4 (maximal coupling) for $B=0.3$ and $C\geqslant 0.7$ $(m_1=m_2=m_3=1)$, and case 3 in all the other cases.

The decrease of the dimension at 0.7 for identical systems is connected with the full synchronization of the systems. Equations (10) admit solutions symmetric in x and u $(x_n-u_n=0)$, which in this region become stable and the whole probability measure is localized on a lower-dimensional manifold. For more details, see [12].

B. Three Hénon maps

Consider now the system U consisting of three Hénon maps [51] coupled as follows [12]:

$$K\begin{cases} x_{i+1}=1.4-x_i^2+0.3y_i\,, \\ y_{i+1}=x_i\,, \end{cases}$$

$$L\begin{cases} u_{i+1}=1.4-[C_1x_i+(1-C_1)u_i]u_i+B_1v_i\,, \\ v_{i+1}=u_i\,, \end{cases} \tag{11}$$

$$M\begin{cases} w_{i+1}=1.4-[C_2x_i+(1-C_2)w_i]w_i+B_2z_i\,, \\ z_{i+1}=w_i\,. \end{cases}$$

Thus Hénon system K drives systems L and M. The coupling is introduced through variables u and w. Parameters C_1 and C_2 measure the strength of interaction.

Suppose the measurements on (K, L, M) yield variables u and w. What can be said in this case about the interaction between the systems L and M?

For $C_1=C_2=0$ neither system L nor M feels the influence of K. They also do not interact (case 1). When one of the C_i grows, the influence of K is immediately mirrored in the rise of the dimension of μ_L or μ_M. For both $C_i>0$ the systems L and M interact (case 2), and the part responsible for interaction is K. Thus the dimension of the common part is constant and equal to 1.22 in our case.

We show this behavior in Fig. 6(a) for nonidentical systems ($B_1=0.3$, $B_2=0.1$) and 7(a) for identical systems ($B_1=0.3$, $B_2=0.3$). In both cases $C_1=0.5$ and C_2 is varied. In both figures one can clearly see the jump of the dimension of

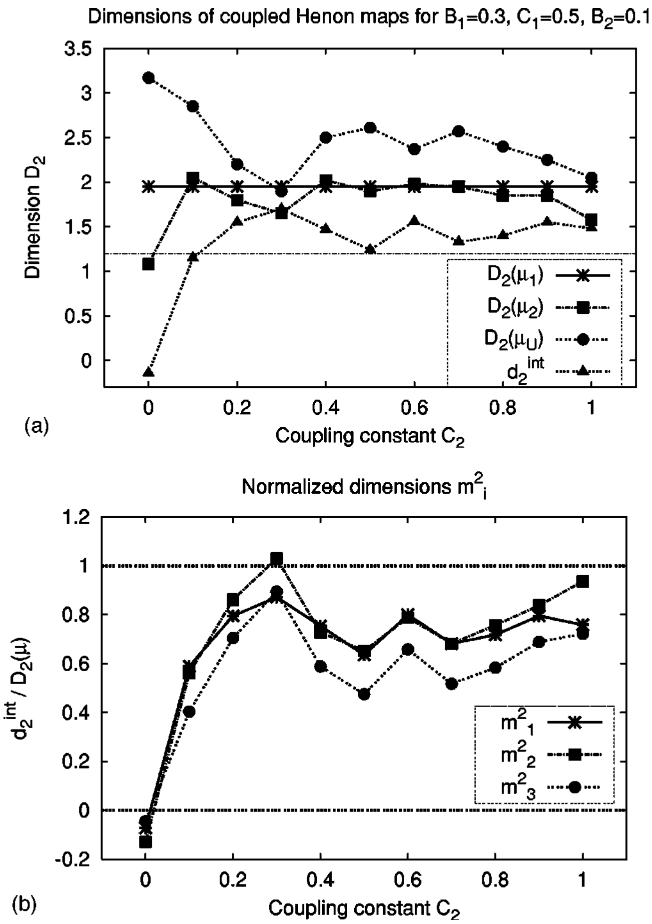

FIG. 6. (a) Dimensions $D_2(\mu_1)$, $D_2(\mu_2)$, $D_2(\mu_U)$, and $d_2^{\rm int}$ of two-way coupled Hénon maps [Eq. (11)] with different response systems ($C_1=0.5$, $B_1=0.3$, $B_2=0.1$). Additional line at 1.2 in the upper figure is a guide for the eye and stands for the dimension of the attractor of Hénon system K. (b) Normalized dimensions m_1^2, m_2^2, and m_U^2 for the same systems.

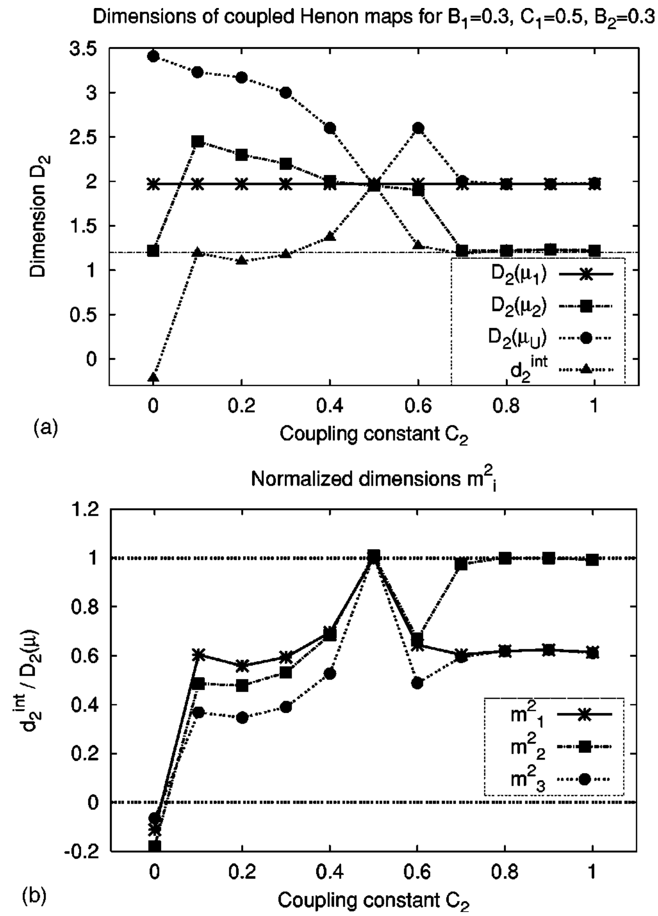

FIG. 7. (a) Dimensions $D_2(\mu_1)$, $D_2(\mu_2)$, $D_2(\mu_U)$, and $d_2^{\rm int}$ of two-way coupled Hénon maps [Eq. (11)] with identical response systems ($C_1=0.5$, $B_1=0.3$, $B_2=0.3$). Additional line at 1.2 in the upper figure is a guide for the eye and stands for the dimension of the attractor of Hénon system K. (b) Normalized dimensions m_1^2, m_2^2, and m_U^2 for the same systems.

interaction from 0 to values equal to or greater than 1.22, the dimension of the attractor of K.

Typical behavior of the local dimension $d \ln C(\epsilon)/d \ln \epsilon$ as a function of resolution ϵ is shown in Fig. 8.

Figure 7 is particularly interesting, since one can apparently identify all four cases of our classification. For $C_2=0$ we have noninteracting systems; for $C_2 \in [0.2,0.4]$ and $C_2=0.6$ we have case 2. For $C_2=0.5$ the two Hénon systems L and M become identical. Since at this value of coupling constant they are in general synchrony with the driver, which means their asymptotic states are independent of their initial states, and depend only on the present state of the driver, it follows that $u_n=w_n$. For $C_2 \geq 0.7$ the system M fully synchronizes with K, which leads to the collapse of the probability measure in K,M space on the diagonal (compare the discussion in the previous subsection).

C. Logistic maps

Let $f_\alpha(x) := \alpha x(1-x)$. Consider a system consisting of four uncoupled logistic maps

$$x_{n+1}^i = f_{\alpha_i}(x_n^i),$$

where $\alpha_1=3.7$, $\alpha_2=3.8$, $\alpha_3=3.9$, and $\alpha_4=4$. Suppose the only variables available experimentally are[4] $Y^{i,j}(n) = F^{i,j}(x_n^i, x_n^j)$, $i<j$. Given two randomly chosen time series $Y^{i,j}(n), Y^{k,l}(n)$ we want to know if they share some degrees of freedom or not (if they ''interact'' or not). If i or j is equal to k or l, there are only three active degrees of freedom in the compound system. Otherwise there are four.

Estimated correlation dimensions for several cases are collected in Table II. In every case we used time series 10^5 points long except for the last one, for which 10^6 points were used. The estimation error was roughly 2% except for the

[4]The coupling functions $F^{i,j}$ were chosen randomly from $x+y$, xy, $\sin(x)\cos(y)$, $x\exp(y)$, and $2x-y$, and fixed.

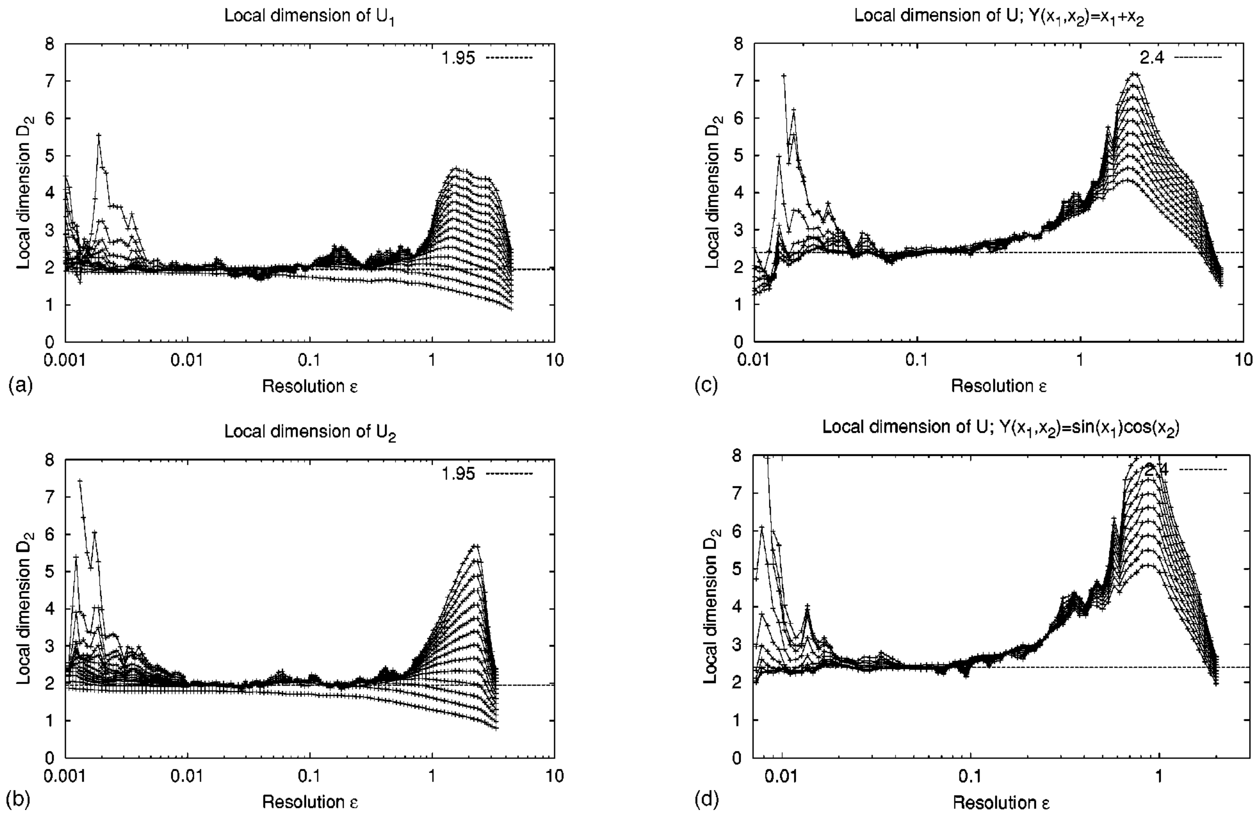

FIG. 8. Local correlation dimension $d \ln C(\epsilon)/d \ln \epsilon$ as a function of resolution ϵ smoothed out with the Takens estimator. Data shown come from two different Hénon systems driven by the third [Eq. (11) with parameters $B_1=0.3$, $C_1=0.5$, $B_2=0.1$, $C_2=0.6$]. Correlation dimension of subsystems U_1 and U_2 estimated from (a) and (b) is 1.95 for both. Dimension of the whole system is estimated to be 2.4 on the basis of five plots for which different coupling functions were used, two of which are shown in (c) and (d). Dimension of interaction d_2^{int} in this case is $1.95+1.95-2.4=1.5>1.22$, which suggests partial synchronization of the two response systems with the driver.

last case for which it was about 5–10 %.[5]

Consider now two symmetrically coupled logistic maps

$$x_{n+1}=f_\alpha(\tilde{x}_n) \quad \text{where } \tilde{x}_n=\frac{x_n+cy_n}{1+c},$$
$$y_{n+1}=f_\beta(\tilde{y}_n), \quad \text{where } \tilde{y}_n=\frac{y_n+cx_n}{1+c}, \tag{12}$$

and the parameter $c\in[0,1]$ measures the coupling. Similar couplings have been discussed previously in the literature (e.g., [54,53]). The maps are uncoupled for $c=0$. For $c=1$ (the strongest coupling) if we set $z_n:=\tilde{x}_n=\tilde{y}_n$, we have $x_n=[2\alpha/(\alpha+\beta)]z_n$, $y_n=[2\beta/(\alpha+\beta)]z_n$, and $z_{n+1}=f_{(\alpha+\beta)/2}(z_n)$. Therefore the dynamics is one dimensional. Case $c>1$ is equivalent to $c'=1/c$.

[5]We believe there are two reasons for this. One is the higher dimensionality of the system in the last case (four uncoupled logistic maps); the other is worse ergodicity in the phase space because the maps are uncoupled. Note that our procedure consists of two parts: first we make the embedding, then we calculate the dimensions. Each of the two can introduce errors. The number expected in the last case is the sum of the first four numbers, namely, 3.87 ± 0.6.

Estimated correlation dimensions for several values of the coupling constant c are shown in Fig. 9. One can see the jump in the dimension of interaction from 0 at $c=0$ to the value equal to the dimension of the whole system for positive c, indicating case 4 in our classification. For $c=0.2$ asymptotic dynamics settles on a period-2 periodic orbit leading to all the dimensions being equal to 0. Numerically obtained approximations to asymptotic measures for the coupling constant $c=0$, 0.1, 0.2, 0.3, 0.4, 0.5 are shown in Fig. 10. Note the increasing synchronization between x and y.

It is of interest to compare the values of dimensions for $c=0$ and 1, because in both cases $D_1(\mu_x)\approx D_1(\mu_y)=1$, but the dimension of the whole system estimated from $f(x_n,y_n)$ is equal to 2 in the first case and 1 in the second, implying $D_q^{\text{int}}=0$ and 1 in these cases, respectively. Thus the first measure has a product structure, while the other is concentrated on the diagonal $x=y$.

The last case considered is that of double control:

$$x_{n+1}=f_\alpha(x_n),$$
$$y_{n+1}=\frac{f_\beta(y_n)+c_1x_n}{1+c_1}, \tag{13}$$
$$z_{n+1}=\frac{f_\gamma(z_n)+c_2x_n}{1+c_2},$$

TABLE II. Estimated correlation dimension for uncoupled logistic maps. The estimation error is roughly 2% except for the last number for which it is about 5–10 %.

Series $x(n)$	$D(\mu_{x(n)})$
x_1	0.96
x_2	0.95
x_3	0.97
x_4	0.99
$Y^{1,2}$	1.88
$Y^{1,3}$	1.94
$Y^{1,4}$	1.95
$Y^{2,3}$	1.89
$Y^{2,4}$	1.94
$Y^{3,4}$	1.93
$f(Y^{1,2},Y^{1,3})$	2.88
$f(Y^{1,2},Y^{3,4})$	3.8

where $\alpha=4.0$, $\beta=3.8$, $\gamma=3.9$, and $c_1,c_2\in[0,1]$. Let the observed systems U_1 and U_2 be the sets of all pairs (x,y) and (x,z), respectively. Then we have essentially case 2. If U_1 and U_2 are the sets of all points x and pairs (x,z) then we have case 3.

Figures 11 and 12 show the estimated correlation dimension in these cases. Again, one can clearly see the difference between the coupled $(c_i>0)$ and uncoupled $(c_i=0)$ systems, because the interaction dimensions jumps from 0 to 1 or more, in agreement with our expectations from Theorems 2 and 3, since the dimension of the common part is 1 (x_n evolves according to the Ulam map: $\alpha=4.0$). Figure 13 shows projections of the attractor of Eq. (13) on the (x,z) and (y,z) planes for $c_1=0.1$ and $c_2=0.2$.

VI. CONCLUSIONS AND OUTLOOK

We have presented a method that allows one to distinguish interacting from noninteracting systems when time series of variables of the two systems are available. Partial proof of its validity was provided. Classification of all possible interaction schemes was presented with examples of all the cases. Several simple interacting systems were analyzed.

To use our method in practice (from field data) we suggest the following procedure.

(A) Calculate the dimensions $D_q(\mu_1)$, $D_q(\mu_2)$, $D_q(\mu_U)$, and d_q^{int} [Eq. (8)] (we suggest $q=1$ or 2; it is also good to normalize the data if they are of different orders). In this respect one might wish to use a coupling function of the type

$$Y\left(\frac{x-\langle x\rangle}{\sigma_x},\frac{y-\langle y\rangle}{\sigma_y}\right).$$

(B) Repeat the calculation for several different coupling functions Y and average the results (linear combination seems to be the best choice).

(C) If they are different from 0, calculate the normalized dimensions m_i^q [Eq. (9)].

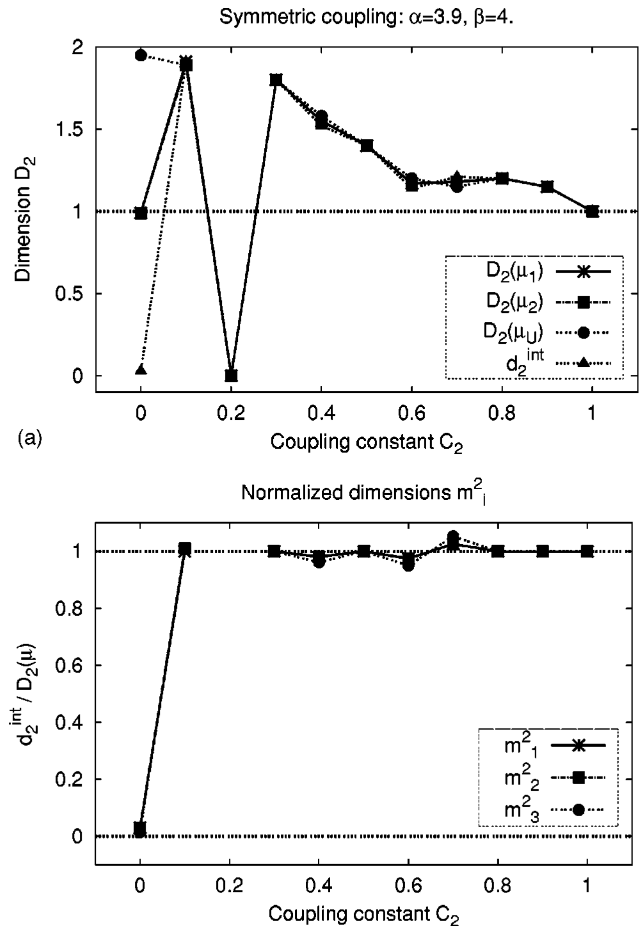

FIG. 9. (a) Dimensions $D_2(\mu_1)$, $D_2(\mu_2)$, $D_2(\mu_U)$, and d_2^{int} of symmetrically coupled logistic maps [Eq. (12)]. (b) Normalized dimensions m_1^2, m_2^2, and m_U^2 for the same systems.

(D) They may take one, two, or three distinct values. (a) If all of them are 0, the systems do not interact (case 1). (b) If all of them are greater than 0 and less than 1, this is a generic case of interacting systems (case 4). (c) If one of them is 1 and the others are smaller, all the degrees of freedom of one system couple to some degrees of freedom of the other (case 3), or we have the previous case (case 4) but the variables of one of the systems that are not coupled to the other synchronize to the system comprising the common part of the dynamics. (d) If they are all equal to 1, all the degrees of freedom of one system couple to all the degrees of freedom of the other (case 2), or we have the two previous cases (3 and 4) but the variables of the two systems that are not coupled synchronize to the system comprising the common part of the dynamics.

Nothing has been said so far about the influence of noise on our method. Put simply, the larger the noise, the more difficult it is to apply. However, noise leads to some interesting phenomena that deserve a longer discussion. This will be provided in our forthcoming paper in which we successfully apply our method to distinguish between interacting and noninteracting Chua systems in an experiment [55]. We hope our method will prove a useful tool in the analysis of other complex systems.

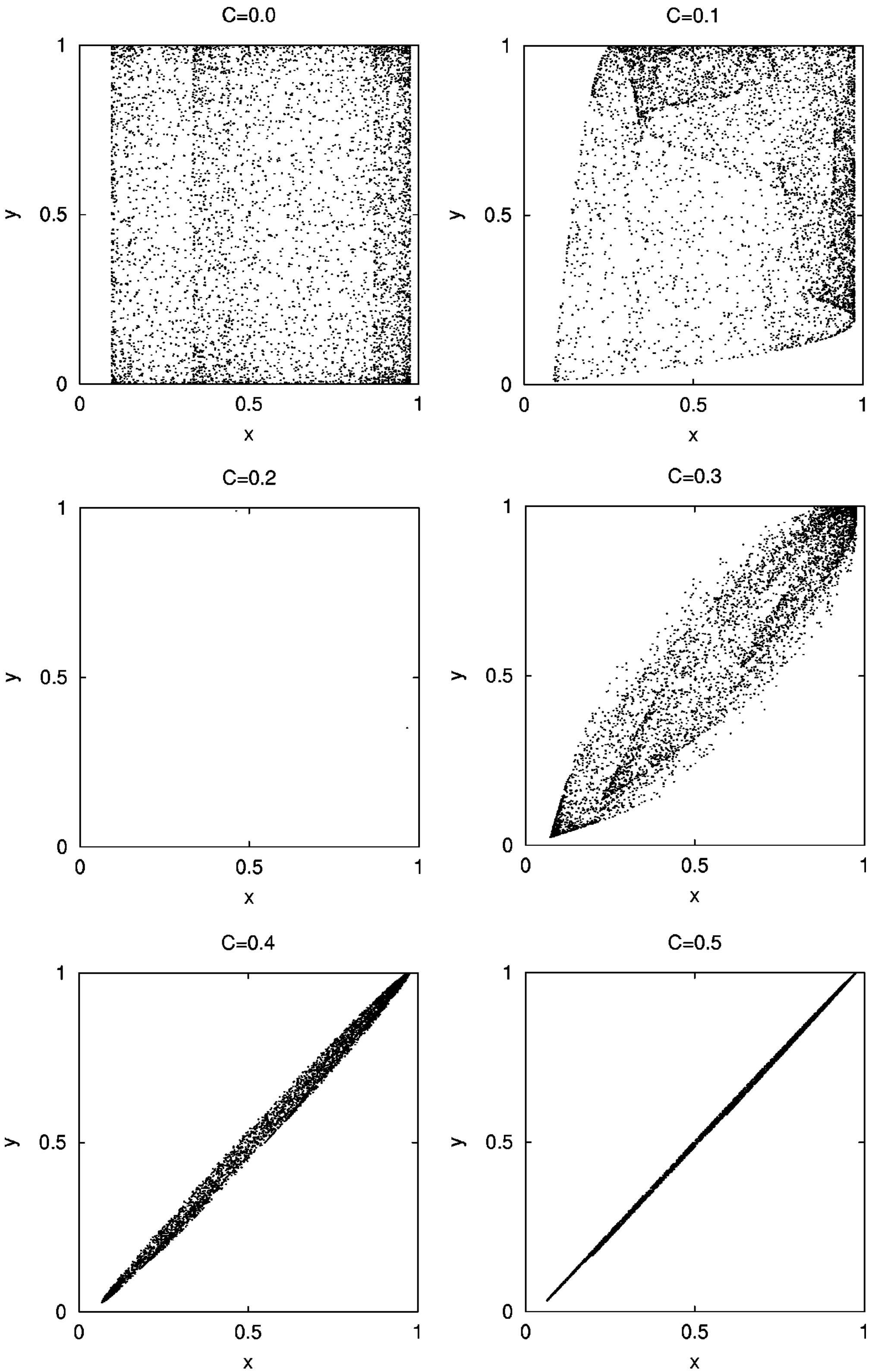

FIG. 10. Attractors of symmetrically coupled logistic maps [Eq. (12)] for $c=0,0.1,0.2,0.3,0.4,0.5$ in the (x,y) plane.

Note added in proof. Ideas similar to some of those presented in this paper were considered in [56] and [57]. They were used, e.g., to distinguish temporal from spatiotemporal chaos in magnetic systems [58]. We thank Dr. Jan Zebrowski for pointing out these references and providing us with them.

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Discussions with several people enriched our understanding of the problem. In particular we want to thank Lou Pecora, Piotr Szymczak, and Karol Życzkowski for illuminating comments. This work was supported by the Polish

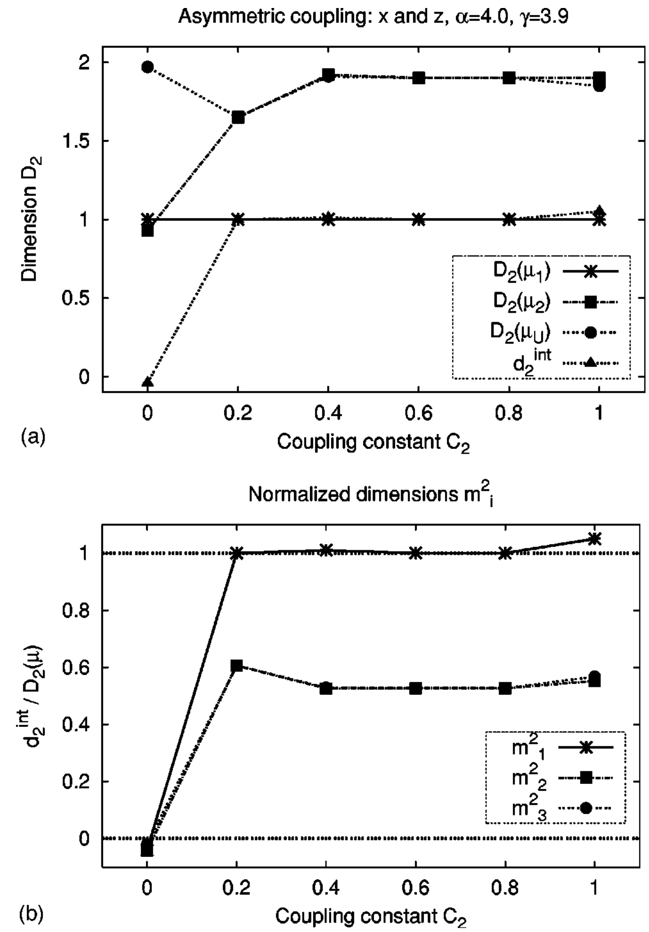

FIG. 11. (a) Dimensions $D_2(\mu_1)$, $D_2(\mu_2)$, $D_2(\mu_U)$, and d_2^{int} of asymmetrically coupled logistic maps [Eq. (13)] when x and z are the observed variables. (b) Normalized dimensions m_1^2, m_2^2, and m_U^2 for the same systems.

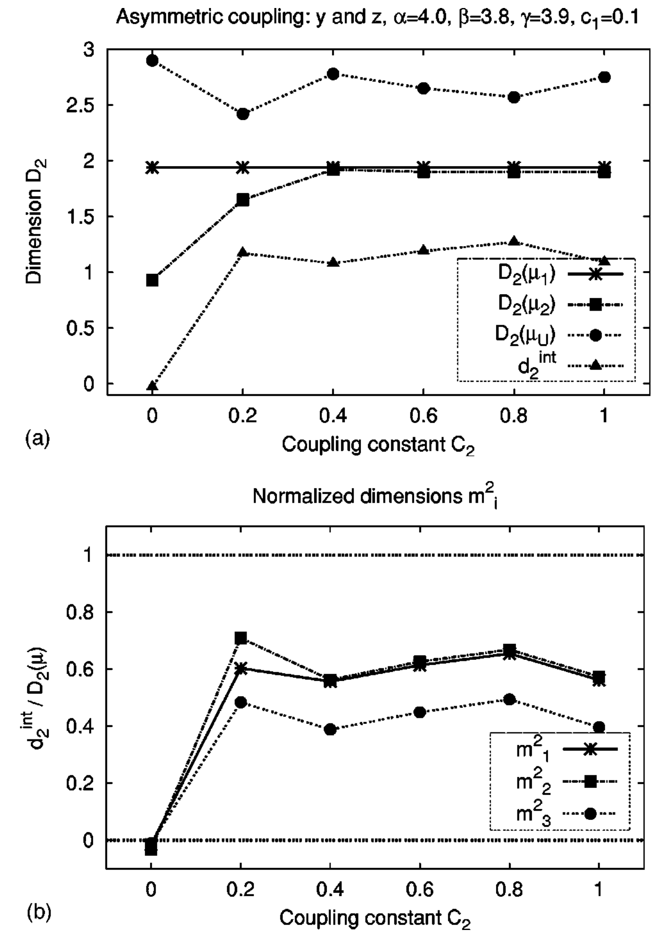

FIG. 12. (a) Dimensions $D_2(\mu_1)$, $D_2(\mu_2)$, $D_2(\mu_U)$, and d_2^{int} of asymmetrically coupled logistic maps [Eq. (13)] when y and z are the observed variables. (b) Normalized dimensions m_1^2, m_2^2, and m_U^2 for the same systems.

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APPENDIX A: THE PROOFS

Let μ_1,μ_2 be the invariant measures of systems U_1,U_2 as defined in Sec. II B.

Theorem 1. Suppose $D_q(\mu_1)$, $D_q(\mu_2)$, and $D_q(\mu_1\times\mu_2)$ exist. Then

$$D_q(\mu_1\times\mu_2)=D_q(\mu_1)+D_q(\mu_2).$$

Proof. Take $q\neq 1$. For every $\epsilon>0$ consider partitions of $\mathbb{R}^{n_i}$ into cells of volume ϵ^{n_i}. This gives a partition in $\mathbb{R}^{n_1+n_2}$ into boxes of volume $\epsilon^{n_1+n_2}$. Let

$$p_j=\mu_1(j\text{th cell from the cover of } U_1),$$

$$r_k=\mu_2(k\text{th cell from the cover of } U_2).$$

Then

$$D_q(\mu_1\times\mu_2)=\lim_{\epsilon\to 0}\frac{1}{q-1}\,\frac{\ln\Sigma_{k,j}p_i^q r_j^q}{\ln\epsilon}$$

$$=\lim_{\epsilon\to 0}\frac{1}{q-1}\,\frac{\ln(\Sigma_k p_i^q)(\Sigma_j r_j^q)}{\ln\epsilon}$$

$$=\lim_{\epsilon\to 0}\left(\frac{1}{q-1}\,\frac{\ln\Sigma_k p_k^q}{\ln\epsilon}\right)+\lim_{\epsilon\to 0}\left(\frac{q}{q-1}\,\frac{\ln\Sigma_j r_j^q}{\ln\epsilon}\right).$$

But the last two limits exist and are equal to $D_q(\mu_1)$ and $D_q(\mu_2)$, respectively.

The case of $q=1$ is straightforward and left to the reader. □

For the next proof we need the following lemma.

Lemma 1. Let $1\geqslant c_{ij}\geqslant 0$, $\Sigma_{ij}c_{ij}=1$, $a_i=\Sigma_j c_{ij}$, and $b_j=\Sigma_i c_{ij}$. Then

$$\sum_{i,j}\,[c_{ij}\ln c_{ij}-a_ib_j\ln(a_ib_j)]\geqslant 0. \tag{A1}$$

Proof. Every convex function f satisfies Jensen's inequality

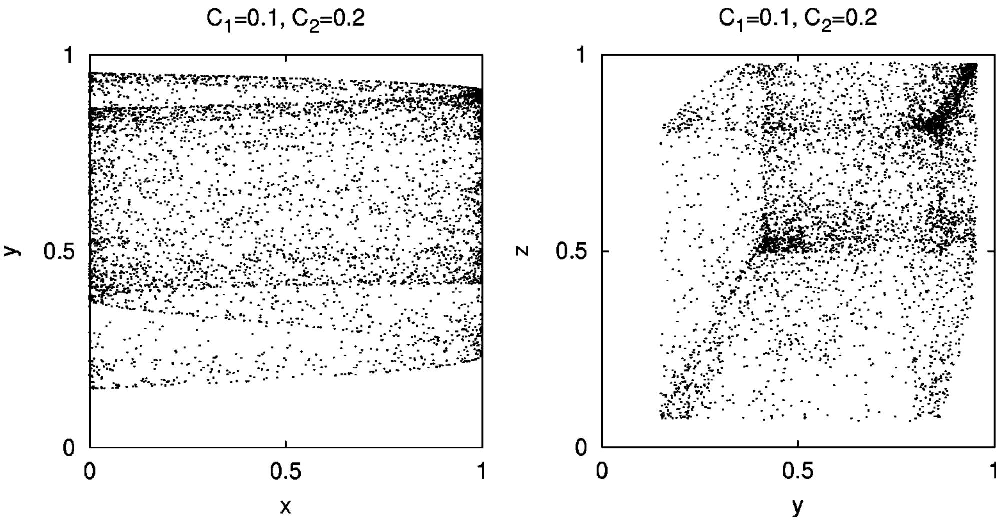

FIG. 13. Projections of the attractor of asymmetrically coupled logistic maps [Eq. (13)] for $c_1=0.1$ and $c_2=0.2$ on the (x,y) and (y,z) planes.

$$f\left(\sum_i p_i x_i\right) \leqslant \sum_i p_i f(x_i), \tag{A2}$$

where $\Sigma_i p_i=1$. Since $f(x)=x\ln x$ is convex, one has

$$f\left(\sum_{i,j} c_{ij}\right) \geqslant \sum_{i,j} a_i b_j f\left(\frac{c_{ij}}{a_i b_j}\right),$$

$$f(1) \leqslant \sum_{i,j} a_i b_j \frac{c_{ij}}{a_i b_j}(\ln c_{ij} - \ln a_i - \ln b_j)$$

$$0 \leqslant \sum_{i,j} c_{ij}\ln c_{ij} - \sum_{i,j} c_{ij}\ln a_i - \sum_{i,j} c_{ij}\ln b_j$$

$$0 \leqslant \sum_{i,j} c_{ij}\ln c_{ij} - \sum_i a_i \ln a_i - \sum_j b_j \ln b_j$$

$$0 \leqslant \sum_{i,j} [c_{ij}\ln c_{ij} - a_i b_j \ln(a_i b_j)],$$

where we used Eq. (A2) with $p_{ij}=a_i b_j$ and $x_{ij}=c_{ij}/(a_i b_j)$. □

Let μ_1, μ_2, μ_V, and μ_U be the invariant measures defined in Sec. II C.

Theorem 2. Suppose $D_1(\mu_1)$, $D_1(\mu_2)$, $D_1(\mu_V)$, and $D_1(\mu_U)$ exist. Then

$$D_1(\mu_V) \leqslant d_{\mathrm{int}} := D_1(\mu_1) + D_1(\mu_2) - D_1(\mu_U).$$

The equality holds when y_1 and y_2 are asymptotically independent.

Proof. Let n_1 be the number of variables $\mathbf{y}_1$ (Sec. II C) with the property that any change in their present state does not influence the future states of system U_2. Let n_2 be the number of variables $\mathbf{y}_2$ of the symmetric property. Let m be the number of all the other variables ($\mathbf{x}$). Variables $\mathbf{x}$ form the part V of the compound system that is responsible for the interaction. Thus the system can be embedded in $\mathbb{R}^{n_1+n_2+m}$.Consider a partition of $\mathbb{R}^{n_1+n_2}$ into cells of size ε consistent with the structure of equations of dynamics, i.e., (i,j,k)th cell$=A_i\times B_j\times C_k$, where A, B, and C are ε cells of dimension, respectively, m, n_1, and n_2 in spaces spanned by $\mathbf{x}$, $\mathbf{y}_1$, and $\mathbf{y}_2$.

Since the dynamics of $(\mathbf{x},\mathbf{y}_1)$ is independent of $\mathbf{y}_2$, the invariant measure $\mu_1(A_i\times B_j)$ can be written as

$$\mu_1(A_i\times B_j) = \mu_V(A_i)\,\mu_{(y_1|x)}(B_j|A_i) =: p_i r_{ji},$$

where $\mu_{(y_1|x)}(B_j|A_i)$ are the conditional probabilities of finding the $\mathbf{y}_1$ in B_j under the condition $\mathbf{x}$ being in A_i. Similarly,

$$\mu_2(A_i\times C_k) = \mu_V(A_i)\,\mu_{(y_2|x)}(C_k|A_i) =: p_i s_{ki}$$

and

$$\mu_S(A_i\times B_j\times C_k) = \mu_V(A_i)\,\mu_{(y_1,y_2|x)}(B_j,C_k|A_i) =: p_i t_{jki}.$$

If $\mu_{(y_1|x)}(B_j|A_i)$ and $\mu_{(y_2|x)}(C_k|A_i)$ are independent, then

$$\mu_{(y_1,y_2|x)}(B_j,C_k|A_i) = \mu_{(y_1|x)}(B_j|A_i)\,\mu_{(y_2|x)}(C_k|A_i); \tag{A3}$$

otherwise the only thing we know is that the left-hand side measure is the coupling of the right-hand side measures, namely,

$$\sum_k \mu_{(y_1,y_2|x)}(B_j,C_k|A_i) = \mu_{(y_1|x)}(B_j|A_i),$$

$$\sum_j \mu_{(y_1,y_2|x)}(B_j,C_k|A_i) = \mu_{(y_2|x)}(C_k|A_i),$$

or

$$\sum_k t_{jki} = r_{ji},$$

$$\sum_j t_{jki} = s_{ki}.$$

Of course,

$$\sum_k s_{ki} = \sum_j r_{ji} = \sum_{jk} t_{jki} = 1$$

if $p_i \neq 0$. Otherwise we take $\forall\, j,k : t_{jki} = 0$.

Taking this into consideration, inequality (5) follows:

$$\begin{aligned} &D_1(\mu_1) + D_1(\mu_2) - D_1(\mu_V) - D_1(\mu_U) \\ &= \lim_{\epsilon\to 0} \frac{\Sigma_i \Sigma_j p_i r_{ji} \ln(p_i r_{ji})}{\ln \epsilon} + \lim_{\epsilon\to 0} \frac{\Sigma_i \Sigma_k p_i s_{ki} \ln(p_i s_{ki})}{\ln \epsilon} - \lim_{\epsilon\to 0} \frac{\Sigma_i p_i \ln(p_i)}{\ln \epsilon} - \lim_{\epsilon\to 0} \frac{\Sigma_{i,j,k} p_i t_{jki} \ln(p_i t_{jki})}{\ln \epsilon} \\ &= \lim_{\epsilon\to 0} \frac{\Sigma_i p_i \ln(p_i)(\Sigma_j r_{ji} + \Sigma_k s_{ki} - 1 - \Sigma_{j,k} t_{jki})}{\ln \epsilon} + \lim_{\epsilon\to 0} \frac{\Sigma_i p_i \Sigma_{j,k} [r_{ji} s_{ki} \ln(r_{ji} s_{ki}) - t_{jki} \ln(t_{jki})]}{\ln \epsilon} \geqslant 0 \end{aligned}$$

where in the last line we used Lemma 1 for $c = t$, $a = r$, and $b = s$ and the fact that $\ln \epsilon < 0$.

Note that the equality holds if and only if

$$t_{jki} = r_{ji} s_{ki}. \tag{A4}$$

This is what we call asymptotical independence of variables y_1 and y_2. In particular, when y_i are in generalized synchrony with x, this means that their asymptotic behavior is independent of their initial states and depends only on the initial state of x; therefore their probability distributions cannot be independent, since they depend on the same number $x(0)$. However, we think this is not the only case where the equality is violated; this is why we use another name for the above condition. □

One would like to establish a similar inequality in the case of other Rényi dimensions; however, in general, even when Eq. (A4) is satisfied,

$$D_q(\mu_S) \neq D_q(\mu_1) + D_q(\mu_2) - D_q(\mu_x).$$

Indeed,

$$\begin{aligned} &D_q(\mu_1) + D_q(\mu_2) - D_q(\mu_x) - D_q(\mu_S) \\ &= \lim_{\epsilon\to 0} \frac{1}{\ln \epsilon} \ln\left[\frac{(\Sigma_{i,k} p_i^q r_{ki}^q)(\Sigma_{l,j} p_l^q s_{jl}^q)}{(\Sigma_l p_l^q)(\Sigma_{i,j,k} p_i^q s_{ji}^q r_{ki}^q)}\right] \\ &= \lim_{\epsilon\to 0} \frac{1}{\ln \epsilon} \ln\left[1 + \frac{\Sigma_{i<i,j,k} p_i^q p_l^q (r_{ki}^q - r_{kl}^q)(s_{ji}^q - s_{jl}^q)}{\Sigma_{i,j,k,l} p_i^q p_l^q s_{ji}^q r_{ki}^q}\right]. \end{aligned} \tag{A5}$$

This may have arbitrary sign and need not vanish in the limit.

Although Eq. (A5) must go to 0 in the limit $q \to 1$, one can perhaps construct examples of measures for which the slope can be arbitrarily large. On the other hand, we believe such measures will not be typically observed in physical systems.

APPENDIX B: AN EXAMPLE OF PARTIALLY COUPLED SYSTEMS

We present here a simple although abstract example of interacting systems for which one can introduce the natural decomposition (4).

Consider two systems U_1, U_2 interacting through a thin contact layer V (Fig. 14). Denote variables in U_1 as $\mathbf{u}_1 = (\mathbf{v}_1, \mathbf{w}_1)$, variables in U_2 as $\mathbf{u}_2 = (\mathbf{v}_2, \mathbf{w}_2)$, and variables of the contact layer V as $(\mathbf{v}_1, \mathbf{v}_2)$. The dynamics of such a configuration can be described as

$$\dot{\mathbf{w}}_1 = f_1(\mathbf{v}_1, \mathbf{w}_1),$$

$$\dot{\mathbf{w}}_2 = f_2(\mathbf{v}_2, \mathbf{w}_2),$$

$$\dot{\mathbf{v}}_1 = g_1(\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}_1),$$

$$\dot{\mathbf{v}}_2 = g_2(\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}_2).$$

If we can average the influence of $\mathbf{w}_1, \mathbf{w}_2$ on the dynamics of $\mathbf{v}_1, \mathbf{v}_2$, e.g., when the time scales involved in the dynamics of $\mathbf{v}_i$ and $\mathbf{w}_i$ are different, we obtain

$$\dot{\mathbf{w}}_1 = f_1(\mathbf{v}_1, \mathbf{w}_1),$$

$$\dot{\mathbf{w}}_2 = f_2(\mathbf{v}_2, \mathbf{w}_2), \tag{B1}$$

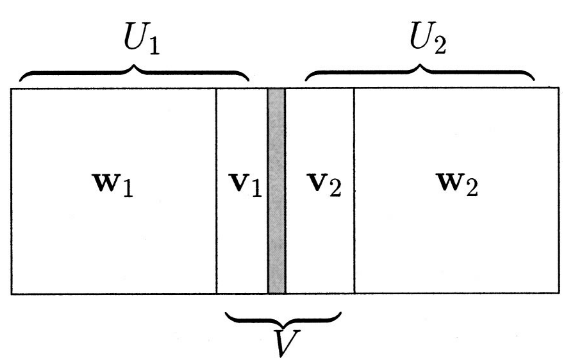

FIG. 14. Interacting systems.

$$\dot{\mathbf{v}}_1 = \tilde{g}_1(\mathbf{v}_1, \mathbf{v}_2, \lambda_1),$$

$$\dot{\mathbf{v}}_2 = \tilde{g}_2(\mathbf{v}_1, \mathbf{v}_2, \lambda_2),$$

where λ_1, λ_2 measure the average influence of $\mathbf{w}_1, \mathbf{w}_2$ on V. Thus equations for $\mathbf{v}_1, \mathbf{v}_2$ comprise a closed system V. This part of the dynamics is responsible for the interaction. Note that this scheme can also be considered as a double control configuration of three systems, where $(\mathbf{v}_1, \mathbf{v}_2)$ control $\mathbf{w}_1$ and $\mathbf{w}_2$.

If we set $\mathbf{x} := (\mathbf{v}_1, \mathbf{v}_2)$, $\mathbf{y}_i = \mathbf{w}_i$, then Eqs. (B1) reduce to Eqs. (4).

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