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Fractality of certain quantum states

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Abstract: We prove the theorem announced in Phys. Rev. Lett. **85**:5022, 2001 concerning the existence and properties of fractal states for the Schrödinger equation in the infinite one-dimensional well.

1. Introduction

Fractals are sets and measures of non-integer dimension [19]. They are good models of phenomena and objects in various areas of science. They are often connected with non-equilibrium problems of growth [22] and transport [11,4]. In this direction their importance can be expected to grow in view of the recent results connecting transport coefficients with fractal properties of hydrodynamic modes [13,12]. Their ubiquity in dynamical systems theory as attractors, repellers, and attractor boundaries is well-known [23,9].

Fractals have also been found in quantum mechanics. For instance, quantum models related to the problem of chaotic scattering often reveal the fractal structure [5,17,3]. Quantum field theories in fractal spacetimes have recently been reviewed in [18].

Recently fractals have been shown to satisfy Schrödinger equations for simplest non-chaotic potentials [1,26]. In this paper we prove some of the results announced in [26].

This work has two objectives. One is to present the first (as far as we know) rigorous proof of fractality of any quantum states. The other is to illustrate a convenient method of calculating dimensions of graphs of continuous functions introduced by Claude Tricot [24].

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One of the oldest fractals is Weierstrass function [25,6]:

$$W(x) = \sum_{n=0}^{\infty} a^n \cos(b^n x\pi). \tag{1}$$

introduced as an example of everywhere continuous nowhere differentiable function by Karl Weierstrass around 1872. Maximum range of parameters for which the above series has required properties was found by Godfrey Harold Hardy in 1916 [14], who also showed that

$$\sup\{|f(x) - f(y)| : |x - y| \leq \delta\} \sim \delta^H, \tag{2}$$

where

$$H = \frac{\ln(1/a)}{\ln b}.$$

From this it easily follows (see below) that the box-counting dimension of the graph of the Weierstrass function $W(x)$ is

$$D_W = 2 + H = 2 + \frac{\ln a}{\ln b} = 2 - \left|\frac{\ln a}{\ln b}\right|. \tag{3}$$

Functions whose graphs have non-integer box-counting dimension will be called *fractal functions*. Even though the box-counting dimension of the Weierstrass function is easy to calculate [24], the proof that its Hausdorff dimension has the same value is still lacking. Lower bounds on the Hausdorff dimension of the graph were found by Mauldin [21,20]. Graphs of random Weierstrass functions were shown to have the same Hausdorff and box-counting dimensions for almost every distribution of phases [16].

The above construction can easily be realized in quantum mechanics. Consider solutions of the Schrödinger equation

$$i\partial_t \Psi(x,t) = -\nabla^2 \Psi(x,t) \tag{4}$$

for a particle in one-dimensional infinite potential well. The general solutions satisfying the boundary conditions $\Psi(0,t) = 0 = \Psi(\pi,t)$ have the form

$$\Psi(x,t) = \sum_{n=1}^{\infty} a_n \sin(nx) e^{-in^2 t}, \tag{5}$$

where

$$a_n = \frac{2}{\pi}\int_0^{\pi} dx\, \sin(nx)\Psi(x,0). \tag{6}$$

Weierstrass quantum fractals are wave functions of the form

$$\Psi_M(x,t) = N_M \sum_{n=0}^{M} q^{n(s-2)} \sin(q^n x) e^{-iq^{2n} t}, \tag{7}$$

where $q = 2, 3, \ldots,$ $s \in (0,2)$.

In the physically interesting case of finite M the wave function Ψ_M is a solution of the Schrödinger equation. The limiting case

$$\Psi(x,t) := \lim_{M\to\infty} \Psi_M(x,t) = N \sum_{n=0}^{\infty} q^{n(s-2)} \sin(q^n x) e^{-iq^{2n}t}, \tag{8}$$

with the normalization constant $N = \sqrt{\frac{2}{\pi}(1 - q^{2(s-2)})}$, is continuous but nowhere differentiable. It is a weak solution of the Schrödinger equation. Note that (8) converges for $|q^{s-2}| < 1 \quad \equiv \quad s < 2$. Later on we will show that the probability density of (8) shows fractal features for $s > 0$. Thus the interesting range of s is $(0, 2)$.

We show that not only the real part of the wave function $\Psi(x,t)$, but also the physically important probability density $P(x,t) := |\Psi(x,t)|^2$ exhibit fractal nature. This is not obvious, because $|\Psi(x,t)|^2$ is the sum of squares of real and imaginary part having usually equal dimensions. One can easily show that the dimension of the graph of a sum of functions whose graph have the same dimensions D can be anything[1] from 1 to D.

Our main results are given by

Theorem 1. *Let $P(x,t)$ denote the probability density of a Weierstrass-like wave function (8). Then*

1. *at the initial time $t = 0$ the probability density, $P_0(x) = P(x,0)$, forms a fractal graph in the space variable (i.e. a* space fractal*) of dimension $D_x = \max\{s, 1\}$;*
2. *the dimension D_x of graph of $P_t(x) = P(x, t = const)$ does not change in time;*
3. *for almost every x inside the well the probability density, $P_x(t) = P(x = \mathrm{const}, t)$, forms a fractal graph in the time variable (i.e. a* time fractal*) of dimension $D_t(x) = D_t := 1 + s/2$;*
4. *for a discrete, dense set of points x_d, $P_{x_d}(t) = P(x_d, t)$ is smooth and thus $D_t(x_d) = 1$;*
5. *for even q the average velocity $\frac{d\langle x\rangle}{dt}(t)$ is fractal with the dimension of its graph equal to $D_v = \max\{(1+s)/2, 1\}$;*
6. *the surface $P(x,t)$ has dimension $D_{xy} = 2 + s/2$.*

The physical meaning of the above theorem has been discussed in [26]. We give its proof in section 3. It is based on the power-law behavior of the average δ-oscillation of the infinite double sum present in $P(x,t) = |\Psi(x,t)|^2$ (55). In section 2 we introduce some fundamental concepts and facts to be used in subsequent analysis. Calculations of probability density and average velocity are provided in the Appendix. Positive real constants are denoted by $c, c_1, c_2, \ldots$

2. Box-counting dimension

In this section we recall several equivalent definitions of box-counting dimension, state a criterion for finding the dimension of a continuous function of one variable

[1] Let f_1 and f_2 be functions with graphs having dimensions, respectively, $1 \le D_1 < D_2 \le 2$. Let $g_1 = f_1 + f_2$, $g_2 = f_1 - f_2$. Then the box-counting dimension of both the graph of g_1 and g_2 is D_2, but the dimension of the graph of their sum $g_1 + g_2 = 2f_1$ is $D_1 \in [1, D_2[$.

and prove a connection between the dimension of graph of function of n variables and the dimensions of its sections. All of this is known with the exception, perhaps, of theorem 3 which might be new. We concentrate on the theory of box-counting dimension for graphs of continuous functions of one variable. More general theory and a deeper presentation can be found for instance in [9, 10, 24].

Let $A \subset \mathbb{R}^n$ be bounded. Consider a grid of n-dimensional boxes of side δ

$$[m_1\delta, (m_1+1)\delta] \times \ldots \times [m_n\delta, (m_n+1)\delta]. \tag{9}$$

Let $N(\delta)$ be the number of these boxes covering the set A. It is always finite because A is bounded.

Definition 1. Box-counting dimension *of the set A is the limit*

$$\dim_B(A) := \lim_{\delta\to 0} \frac{\ln N(\delta)}{\ln 1/\delta}. \tag{10}$$

If the limit does not exist one considers upper *and* lower box-counting dimensions

$$\overline{\dim}_B(A) := \limsup_{\delta\to 0} \frac{\ln N(\delta)}{\ln 1/\delta}, \tag{11}$$

$$\underline{\dim}_B(A) := \liminf_{\delta\to 0} \frac{\ln N(\delta)}{\ln 1/\delta} \tag{12}$$

which always exist and satisfy

$$\overline{\dim}_B(A) \geq \underline{\dim}_B(A). \tag{13}$$

The box-counting dimension exists if the upper and lower box-counting dimensions are equal.

Several equivalent definitions are in use (see [19, 9, 10, 24] for a review). The most convenient definition to study the fractal properties of graphs of continuous functions is given in terms of δ-variations [24]. It is essentially a variant of Bouligand definition [2]. We shall restrict our attention to dimensions of curves being subsets of a plane.

Let $K_\delta(x)$ be a closed ball $\{y \in \mathbb{R}^2 : |x-y| \leq \delta\}$.

Definition 2. Minkowski sausage *or* δ-parallel body *of $A \subset \mathbb{R}^2$ is*

$$A_\delta := \bigcup_{x\in A} K_\delta(x) \tag{14}$$

$$= \{y \in \mathbb{R}^2 : \exists x \in A,\ |x-y| \leq \delta\}. \tag{15}$$

Thus the Minkowski sausage of A is the set of all the points located within δ of A.

Proposition 1. *The* box-counting dimension *of a set $A \subset \mathbb{R}^2$ satisfies*

$$\dim_B(A) = \lim_{\delta\to 0}\left(2 - \frac{\ln V(A_\delta)}{\ln \delta}\right), \tag{16}$$

where $V(\delta) = \mathrm{vol}^2(A_\delta)$ is the area of the Minkowski sausage of A.

Proof. Every square from the δ-grid containing $x \in A$ is included in $K_{\sqrt{2}\delta}(x)$. On the other hand, every closed ball of radius $\sqrt{2}\delta$ can be covered by at most 16 squares from the grid. Therefore

$$\delta^2 N(\varepsilon) \le V(A_{\sqrt{2}\delta}) \le 16\delta^2 N(\varepsilon). \tag{17}$$

□

Consider a continuous function on a closed interval $f : [a, b] \to \mathbb{R}$. Its graph is a curve in the plane. To find its box-counting dimension estimate the number of boxes $N(\delta)$ intersecting the graph.

Choose column $\{(x, y) : x \in [n\delta, (n+1)\delta]\}$. Since the curve is continuous the number of the boxes in this column intersecting the graph of f is at least

$$\left\lfloor \sup_{x\in[n\delta,(n+1)\delta]} f(x) - \inf_{x\in[n\delta,(n+1)\delta]} f(x) \right\rfloor / \delta \tag{18}$$

and no more than the same plus 2. If f was a record of a signal then the difference between the maximum and minimum value of f on the given interval quantifies how the signal oscillates on this interval. That's why it is called *oscillation.*

Definition 3. δ-oscillation *of f at x is*

$$\begin{aligned}
\operatorname{osc}_\delta(x)(f) &:= \sup_{|y-x|\le\delta} f(y) - \inf_{|y-x|\le\delta} f(y) && (19)\\
&= \sup\{|f(y) - f(z)| \,:\, y, z \in [a, b] \cap [x - \delta, x + \delta]\}. && (20)
\end{aligned}$$

We will skip (f) if it is clear from the context which function we consider.

From (18) we obtain the following estimate on the total number of boxes covering the graph of f:

$$\sum_{m=1}^{M} \operatorname{osc}_{\delta/2}(x_m)/\delta \le N(\delta) \le 2M + \sum_{m=1}^{M} \operatorname{osc}_{\delta/2}(x_m)/\delta, \tag{21}$$

where $x_m = a + (m - 1/2)\delta$ is the middle of the m–th column from the cover of the graph and $M = \lceil \frac{b-a}{\delta} \rceil$ is the number of columns in the cover ($\lceil x \rceil$ stands for the smallest integer greater or equal to x). Thus

$$N(\delta) \approx M\, \overline{\operatorname{osc}}_{\delta/2}/\delta. \tag{22}$$

If the graph of f has the box-counting dimension D, $N(\delta)$ scales as δ^{-D}. This implies the following scaling of the oscillations

$$\overline{\operatorname{osc}}_{\delta/2} \approx N(\delta)\delta/M \propto \delta^{2-D}. \tag{23}$$

We have thus suggested a connection between the box-counting dimension of the graph and the scaling exponent of the average oscillation of the function f.

Definition 4. δ-variation *of function f is*

$$\begin{aligned}
\operatorname{Var}_\delta(f) &:= \int_a^b \operatorname{osc}_\delta(x)(f)\, dx && (24)\\
&=: (b - a)\overline{\operatorname{osc}}_\delta(f). && (25)
\end{aligned}$$

Geometrically, variation is the area of the set scanned by the graph of f moved horizontally $\pm\delta$ and truncated at $x = a$ and $x = b$, thus it is a kind of Minkowski sausage constructed with horizontal intervals of length 2δ. This observation leads to a convenient technique for calculating dimensions.

Theorem 2. *Let $f(x)$ be a non constant continuous function on $[a, b]$, then*

$$\dim_B \mathrm{graph} f = \lim_{\delta\to 0}\left(2 - \frac{\ln \mathrm{Var}_\delta(f)}{\ln \delta}\right). \tag{26}$$

The proof consists of showing equivalence of $\mathrm{Var}_\delta(f)$ with the Minkowski sausage and follows from inequality ([24], p. 130–132, 148–149)

$$\mathrm{Var}_\delta(f) \le V(A_\delta) \le c\mathrm{Var}_\delta(f), \tag{27}$$

where

$$\begin{aligned} A &= \mathrm{graph} f \\ c &= c_1 + c_2/s \\ s &= \left[\sup_{x\in[a,b]} f(x) - \inf_{x\in[a,b]} f(x)\right]. \end{aligned}$$

This is where the assumption of non-constancy of f comes in. Derivation of (27) is not difficult but rather lengthy and will be omitted.

This theorem is the main tool to prove Theorem 1. In order to find the dimensions we will look for estimates of δ-variation. They will usually take the following form:

Proposition 2.

1. $\mathrm{osc}_\delta(x) f(x) \le c\delta^{2-s} \Rightarrow \dim_B \mathrm{graph} f \le s$.
2. $W := \int_a^b |f(x+\delta) - f(x-\delta)| dx \ge c\delta^{2-s} \Rightarrow \dim_B \mathrm{graph} f \ge s$.

Proof.

1. $\mathrm{Var}_\delta f = \int_a^b \mathrm{osc}_\delta(x)(f) dx \le (b-a)c\delta^{2-s}$.
2. $\mathrm{osc}_{2\delta}(x) f \ge |f(x+\delta) - f(x-\delta)| \Rightarrow \mathrm{Var}_\delta f \ge (b-a)c(\delta/2)^{2-s}$.

□

To prove the last point of theorem 1 we need to know what is the dimension of the graph of $f : \mathbb{R}^n \to \mathbb{R}$ given all the dimensions of its one-variable restrictions.

Theorem 3. *Let $f \in \mathcal{C}^0([a_1, b_1]\times\ldots\times[a_n, b_n])$. For every point $x = (x^1, \ldots, x^n) \in [a_1, b_1] \times \ldots \times [a_n, b_n]$ define $\tilde{x}^i := (x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^n)$. Then*

$$f_i[\tilde{x}^i_0](x^i) := f(x^1_0, \ldots, x^{i-1}_0, x^i, x^{i+1}_0, \ldots, x^n_0) \tag{28}$$

is a restriction of f to a line parallel to i-th axis going through x_0 and $f_i[\tilde{x}^i_0] \in \mathcal{C}^0([a_i, b_i])$.

1. If $\forall x : \mathrm{osc}_\delta f_i[\tilde{x}^i] \le c_i\delta^{H_i}$ then $\dim_B \mathrm{graph} f(x^1, \ldots, x^n) \le n+1-\min\{H_1, \ldots, H_n\}$.

2. *If* $\mathrm{Var}_\delta f_i[\tilde{x}_0^i] \geq c_i \delta^{H_i}$ *for a dense set* $\tilde{x}_0^i \in A \subset \overline{A} = [a_1, b_1]\times\ldots\times[a_{i-1}, b_{i-1}]\times [a_{i+1}, b_{i+1}]\times\ldots\times[a_n, b_n]$ *then* $\dim_B \mathrm{graph} f(x^1, \ldots, x^n) \geq n+1-\min\{H_1, \ldots, H_n\}$.
3. *If all of the above conditions are satisfied then*

$$\begin{aligned}\dim_B \mathrm{graph} f(x^1, \ldots, x^n) &= n + 1 - \min\{H_1, \ldots, H_n\} \\ &= n - 1 + \max\{s_1, \ldots, s_n\}\end{aligned}$$

where

$$s_i = \sup_{\tilde{x}^i} \dim_B \mathrm{graph} f_i[\tilde{x}^i](x^i).$$

In other words, the strongest oscillations along any direction determine the box-counting dimension of the whole $n+1$-dimensional graph.

Proof. We will show the theorem for $n = 2$ for notational simplicity. Generalization to arbitrary n is immediate. Let $f : [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}$. Divide the domain into squares $X_i \times Y_j$ of side δ. This gives rise to K columns A_{ij} of δ-grid in $\mathbb{R}^3$, $1 \leq \frac{K\delta^2}{(b_1-a_1)(b_2-a_2)} \leq 2$.

1. The number of δ-cubes having a common point with the graph of f in column A_{ij} is not greater than $(\sup_{A_{ij}} f - \inf_{A_{ij}} f)/\delta + 2$. But

$$\begin{aligned}|f(x_1, y_1) - f(x_2, y_2)| &= |f(x_1, y_1) - f(x_1, y_2) + f(x_1, y_2) - f(x_2, y_2)| \\ &\leq |f(x_1, y_1) - f(x_1, y_2)| + |f(x_1, y_2) - f(x_2, y_2)|.\end{aligned}$$

Therefore

$$\begin{aligned}\sup_{A_{ij}} f - \inf_{A_{ij}} f &= \sup_{(x_1,y_1),(x_2,y_2)\in A_{ij}} |f(x_1, y_1) - f(x_2, y_2)| \\ &\leq \sup_{x\in X_i} \sup_{y\in Y_j} f(x, y) + \sup_{y\in Y_j} \sup_{x\in X_i} f(x, y) \\ &\leq \sup_{x\in X_i} \mathrm{osc}_{\delta/2} f_1[x] + \sup_{y\in Y_j} \mathrm{osc}_{\delta/2} f_2[y] \\ &\leq c\delta^{\min\{H_1,H_2\}}.\end{aligned}$$

Thus

$$\begin{aligned}\dim_B \mathrm{graph} f(x^1, x^2) &\leq \lim_{\delta\to 0} \frac{\ln(Kc\delta^{\min\{H_1,H_2\}}/\delta)}{\ln 1/\delta} \\ &\leq 3 - \min\{H_1, H_2\}.\end{aligned}$$

2. Set $x \in X_i$. From (17) and (27) it follows that the number $N_i(\delta)$ of δ-cubes in columns A_{ij} covering the graph of $f_2[x](y)$ and the variation of f_2 satisfy

$$\mathrm{Var}_\delta f_2[x] \leq c\delta^2 N_i(\delta). \tag{29}$$

Thus

$$N_i(\delta) \geq c \sup_{x\in X_i} \mathrm{Var}_\delta f_2[x]/\delta^2 \geq c\delta^{H_2-2}. \tag{30}$$

Therefore the number $N(\delta)$ of boxes covering the whole graph of f satisfies

$$N(\delta) \geq c \sum_{i=1}^{M} \sup_{x\in X_i} \mathrm{Var}_\delta f_2[x]/\delta^2 \geq c_1\delta^{H_2-3}. \tag{31}$$

The same can be repeated for any direction thus

$$N(\delta) \geq c_2 \delta^{\min\{H_1,H_2\}-3}. \tag{32}$$

3. An immediate corollary.

Generalization to arbitrary n is achieved by observing that $K\delta^n \approx$ const. $\square$

Remark 1. Another definition which has some convenient technical properties is the Hausdorff dimension [15,7,8], however it is often too difficult to calculate. For instance, as mentioned before, there is still no proof that the Hausdorff dimension of the Weierstrass function is equal to its box-counting dimension. Thus in practice one usually uses the (upper) box-counting dimension. This is also our present approach. It is often assumed that the box-counting dimension and the Hausdorff dimension are equal. A general characterization of situations when this conjecture really holds is also lacking.

3. Proof of Theorem 1

1., 2. We will show that for every fixed t the graph of the probability density $|\Psi|^2$ (55) as a function of x has the box-counting dimension s.

(a) Fix t. Let

$$P_n(x) := \sum_{k=0}^{n} q^{k(s-2)} \sum_{l=0}^{k} \sin q^l x \sin q^{k-l} x \cos(q^{2l} - q^{2(k-l)})t,$$

$q = 2, 3, \ldots$. It is a smooth function whose derivative at every point satisfies

$$\begin{aligned} |P_n'(x)| &\leq 2 \sum_{k=0}^{n} q^{k(s-2)} \sum_{l=0}^{k} q^l |\cos q^l x \sin q^{k-l} x| \\ &\leq 2 \sum_{k=0}^{n} q^{k(s-2)} \frac{q^{k+1}}{q-1} \leq d_1(s,q) q^{n(s-1)}, \end{aligned}$$

where

$$d_1(s,q) = \frac{2q^s}{(q-1)(q^{s-1}-1)}.$$

Let $\delta = q^{-n}$. Then

$$\operatorname{osc}_\delta(x) P_n \leq 2\delta \sup_{x\in[0,\pi]} |P_n'(x)| \leq 2 d_1(s,q) \delta^{2-s}. \tag{33}$$

On the other hand, for

$$\begin{aligned} R_n(x) &:= P(x) - P_n(x) \\ &= \sum_{k=n+1}^{\infty} q^{k(s-2)} \sum_{l=0}^{k} \sin q^l x \sin q^{k-l} x \cos(q^{2l} - q^{2(k-l)})t \end{aligned}$$

we have

$$\mathrm{osc}_\delta(x) R_n \leq 2 \sum_{k=n+1}^{\infty} q^{k(s-2)}(k+1) \leq \frac{4q^{(n+1)(s-2)}n}{(1-q^{s-2})^2}.$$

Polynomial growth is slower than exponential, therefore for arbitrarily small ε there is some M such that

$$\forall n > M : \quad n < (q^\varepsilon)^n.$$

This leads to the following estimate of the oscillation of R_n:

$$\mathrm{osc}_\delta(x) R_n \leq d_2(s,q)\delta^{2-s-\varepsilon},$$

where

$$d_2(s,q) = \frac{4q^{s-2}}{(1-q^{s-2})^2}.$$

Thus for all x and $\delta = q^{-n}$, where $(\ln n)/n < \varepsilon \ln q$, we have

$$\begin{aligned}\mathrm{osc}_\delta(x) P &\leq \mathrm{osc}_\delta(x) P_n + \mathrm{osc}_\delta(x) R_n \\ &\leq (2d_1 + d_2)\delta^{2-s-\varepsilon}.\end{aligned}$$

From proposition 2 it follows that

$$\dim_B \mathrm{graph}\, P_t(x) \leq 2 - (2-s-\varepsilon) = s + \varepsilon \xrightarrow[\varepsilon \to 0]{} s.$$

(b) Fix t. Let $f(x) = P(x,t)$. We want to show that

$$W := \int_a^b |f(x+\delta) - f(x-\delta)| dx \geq c\delta^{2-s}.$$

Take $a = 0$, $b = \pi$. Notice that (we skip the normalization constant)

$$\begin{aligned} W &= \int_0^\pi |f(x+\delta) - f(x-\delta)| dx \qquad (34) \\ &= \int_0^\pi \left| \sum_{k=0}^\infty q^{k(s-2)} \sum_{l=0}^k \{\sin q^l(x+\delta) \sin q^{k-l}(x+\delta) + \right. \\ &\quad \left. - \sin q^l(x-\delta) \sin q^{k-l}(x-\delta)\}\, a_{kl} \right| dx \\ &= \int_0^\pi \left| \sum_{k=0}^\infty q^{k(s-2)} \sum_{l=0}^k \{\cos q^l x \sin q^{k-l} x \sin q^l \delta \cos q^{k-l}\delta\}\, a_{kl} \right| dx, \end{aligned}$$

where

$$a_{kl} = \cos(q^{2k} - q^{2l})t.$$

Take $|h(x)| \leq 1$. Observe, that

$$\begin{aligned} \int_a^b \left|\sum_i f_i(x)\right| dx &\geq \int_a^b |h(x)| \left|\sum_i f_i(x)\right| dx \\ &\geq \left|\int_a^b \sum_i h(x) f_i(x) dx\right| \\ &\geq \left|\int_a^b h(x) f_k(x) dx\right| - \sum_{i \neq k} \left|\int_a^b h(x) f_i(x) dx\right| . \end{aligned} \tag{35}$$

One can interchange the order of summation and integration because $f(x)$ is absolutely convergent. Let us take $\delta = q^{-N}$, $h(x) = \sin q^m x \cos q^n x$. After substitution in (34) using (35) we obtain

$$\begin{aligned} W &\geq \left|\sum_{k=0}^{\infty} q^{k(s-2)} \sum_{l=0}^{k} \sin q^{l-N} \sin q^{k-l-N} \cdot \right. \\ &\quad \left. \cdot \int_0^{\pi} dx \, \sin q^l x \cos q^{k-l} x \cos q^m x \cos q^n x \, a_{kl} \right| \\ &= \frac{\pi}{4} q^{(m+n)(s-2)} \left|\cos q^{m-N} \sin q^{n-N} \cos(q^{2(m+n)} - q^{2m}) t\right| \\ &=: \frac{\pi}{4} \widetilde{W} \end{aligned}$$

In what follows we will show three consecutive proofs that

$$\exists c: \quad \widetilde{W} = q^{(m+n)(s-2)} \left|\cos q^{m-N} \sin q^{n-N} \cos(q^{2(m+n)} - q^{2m}) t\right| \geq c q^{N(s-2)},$$

for $t = \frac{k\pi}{q^l}$, $t/\pi \in \mathbb{Q}$, and a general proof for arbitrary real t. We will take advantage of the fact that q is integer.
Let $N = m + n$. Then

$$\widetilde{W} = q^{N(s-2)} \left|\cos q^{m-N} \sin q^{-m} \cos(q^{2N} - q^{2m}) t\right| .$$

It is enough to consider $t \in [0, \pi[$.

i. Let $t = \frac{k\pi}{q^l}$. Take m such that $2m \geq l$, for instance $m = l$. Then

$$(q^{2N} - q^{2m}) \frac{k\pi}{q^l} = (q^{2N-l} - q^l) k\pi,$$

therefore

$$\left|\cos\left[(q^{2N} - q^{2l}) \frac{k\pi}{q^l}\right]\right| = 1.$$

We also have $\sin q^{-l} = \text{const}$ and

$$\cos 1 = \cos q^0 \leq \cos q^{l-N} \leq \cos q^{-\infty} = \cos 0.$$

This means that for $t = \frac{k\pi}{q^l}$, for sufficiently large N

$$\widetilde{W} \geq q^{N(s-2)} \cos 1 \cdot \sin q^{-l} \cdot \cos 0 = \text{const } q^{N(s-2)}.$$

ii. Let $t/\pi \in \mathbb{Q}$. For the next two proofs let us write t/π in q basis

$$\frac{t}{\pi} = \frac{a_1}{q} + \frac{a_2}{q^2} + \frac{a_3}{q^3} + \ldots = \sum_{k=1}^{\infty} \frac{a_k}{q^k}, \tag{36}$$

where $a_k \in \{0, 1, \ldots, q-1\}$. Therefore

$$\begin{aligned}
&\cos[(q^{2N} - q^{2m})t] \\
&= \cos[\pi(q^{2N-1}a_1 + q^{2N-2}a_2 + \ldots + a_{2N} + q^{-1}a_{2N+1} + \ldots + \\
&\quad + q^{2m-1}a_1 + q^{2m-2}a_2 + \ldots + a_{2m} + q^{-1}a_{2m+1} + \ldots)] \\
&= \cos\left[\pi\left(\frac{a_{2N+1} - a_{2m+1}}{q} + \frac{a_{2N+2} - a_{2m+2}}{q^2} + \ldots\right)\right].
\end{aligned} \tag{37}$$

If we could only choose m so that the first two terms in this series cancel out, we would have a lower estimate on the cosine, because in this case

$$\left|\frac{a_{2N+3} - a_{2m+3}}{q^3} + \ldots\right| \leq (q-1)\left(\frac{1}{q^3} + \frac{1}{q^4} + \ldots\right) = \frac{1}{q^2}.$$

Thus

$$\cos[(q^{2N} - q^{2m})t] \geq \cos(\pi/q^2) \geq \cos\pi/4.$$

Take a rational t. In any basis q the expansion (36) of t/π is finite or periodic. Finite expansions have been treated in the previous point, thus we assume here the expansion of t/π is periodic with period T. Thus for sufficiently large n we have

$$a_{n+T} = a_n$$

so that t/π can be written as

$$t/\pi = 0.a_1 a_2 \ldots a_K (a_{K+1} \ldots a_{K+T}),$$

where $(a_{K+1} \ldots a_{K+T})$ means "repeat $a_{K+1} \ldots a_{K+T}$ periodically ad infinitum". We shall now estimate $\widetilde{W}$ for this t. Every $N > K$ can be written as $N = K + pT + r$, where p is natural or 0 and $r \in \{1, 2, \ldots, T\}$. If we now take $m = K + r$ then not only the first two but all the terms in (37) cancel out, therefore

$$\cos[(q^{2N} - q^{2m})t] = 1.$$

Obviously,

$$\begin{aligned}
\sin q^{-m} &\geq \sin q^{-(K+T)}, \\
\cos q^{m-N} &= \cos q^{-pT} \geq \cos 1,
\end{aligned}$$

Therefore

$$\widetilde{W} \geq q^{N(s-2)} \sin q^{-(K+T)} \cos 1 = \text{const } q^{N(s-2)}.$$

iii. General case of an arbitrary $t/\pi \in [0,1]$. Its expansion in q basis is given by (36). Let A be the set of all the two element sequences with elements from the set $\{0,1,\ldots,q-1\}$. Thus

$$A = \{\{0,0\},\{0,1\},\ldots,\{0,q-1\},\{1,0\},\ldots,\{q-1,q-1\}\},$$

we write $A_{k,l} := \{k,l\}$, $k,l \in \{0,1,\ldots,q-1\}$. Consider all the pairs of consecutive q-digits of t/π of the form

$$\{a_{2m+1}, a_{2m+2}\}, \tag{38}$$

i.e. $\{a_1,a_2\},\{a_3,a_4\}$, etc. Every such pair is equal to some $A_{k,l}$. Let $N_{k,l}$ be the first such m for which

$$A_{k,l} = \{a_{2m+1}, a_{2m+2}\}.$$

If $A_{k,l}$ for given k,l doesn't appear in the sequence of all the pairs (38) we set $N_{k,l} = 0$. Let

$$M = \sup_{k,l} N_{k,l}.$$

Thus if $n > M$ the sequence $\{a_{2n+1}, a_{2n+2}\}$ has appeared at least once among the pairs $\{a_1,a_2\},\{a_3,a_4\},\ldots,\{a_{2M+1},a_{2M+2}\}$. Therefore, for every $N > M$ we can find such an $m \in 1,2,\ldots,M$ that

$$\left|\cos\left[(q^{2N}-q^{2m})\frac{t}{\pi}\pi\right]\right| \geq \cos\frac{\pi}{q^2} \geq \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}.$$

Also

$$\sin q^{-m} \geq \sin q^{-M},$$
$$\cos q^{m-N} \geq \cos q^{M-N} \geq \cos 1,$$

which leads to

$$\widetilde{W} \geq q^{N(s-2)}\frac{\sqrt{2}}{2}\sin q^{-M}\cos 1 = \text{const}\ q^{N(s-2)}.$$

We have thus shown that for every t, for natural q and for $\delta = q^{-N}$

$$W \geq \text{const}\ \delta^{2-s},$$

therefore (proposition 2)

$$\dim_B \operatorname{graph} P_t(x) \geq 2-(2-s) = s.$$

3. *For almost every x, $D_t(x) = \dim_B \operatorname{graph} P_x(t) = D_t := 1 + s/2$.*

We will use the form (56) of the probability density. It is enough to analyze the dimension of

$$\widetilde{P}(t) := \sum_{c=1}^{\infty} q^{2c(s-2)} \sin q^c x \sum_{d=1}^{c} q^{-d(s-2)} \sin q^{c-d} x \cos[(q^{2c} - q^{2(c-d)})t].$$

(a) Let

$$P_n(t) := \sum_{c=1}^{n} q^{2c(s-2)} \sin q^c x \sum_{d=1}^{c} q^{-d(s-2)} \sin q^{c-d} x \cos[(q^{2c} - q^{2(c-d)})t].$$

Then

$$\begin{aligned}
|P_n'(t)| &= \left| \sum_{c=1}^{n} q^{2c(s-2)} \sin q^c x \sum_{d=1}^{c} q^{-d(s-2)} \sin q^{c-d} x \cdot \right. \\
&\quad \left. \cdot (q^{2c} - q^{2(c-d)}) \sin[(q^{2c} - q^{2(c-d)})t] \right| \\
&\le \sum_{c=1}^{n} q^{2c(s-2)} \sum_{d=1}^{c} q^{-d(s-2)} (q^{2c} - q^{2(c-d)}) \\
&= \sum_{c=1}^{n} q^{2c(s-2+1)} \sum_{d=1}^{c} q^{-d(s-2)} (1 - q^{-2d}) \\
&= \frac{q^{2-s}}{q^{2-s}-1} \left[q^s \frac{q^{ns}-1}{q^s-1} - q^{2(s-1)} \frac{q^{2n(s-1)}-1}{q^{2(s-1)}-1} \right] + \\
&\quad - \frac{q^{-s}}{q^{-s}-1} \left[q^{s-2} \frac{q^{n(s-2)}-1}{q^{s-2}-1} - q^{2(s-1)} \frac{q^{2n(s-1)}-1}{q^{2(s-1)}-1} \right].
\end{aligned}$$

Therefore, for n large enough

$$|P_n'(t)| \le c_1 q^{n \max\{s, 2(s-1), s-2\}} = c q^{ns}.$$

Let $\delta = q^{-\alpha n}$. Then $q^n = \delta^{-1/\alpha}$ and

$$\mathrm{osc}_\delta(t) P_n \le c_1 2\delta q^{ns} = 2c_1 \delta^{1-s/\alpha}.$$

Let

$$R_n(t) := \widetilde{P}(t) - P_n(t).$$

Then

$$\begin{aligned}
\mathrm{osc}_\delta(t) R_n &\le 2 \sum_{c=n+1}^{\infty} q^{2c(s-2)} \sum_{d=1}^{c} q^{-d(s-2)} \\
&\le \frac{2q^{2-s}}{q^{2-s}-1} \sum_{c=n+1}^{\infty} q^{2c(s-2)+c(2-s)} \\
&= c_2 \delta^{(s-2)/\alpha}.
\end{aligned}$$

To obtain consistent estimate we must set

$$1 - \frac{s}{\alpha} = \frac{2}{\alpha} - \frac{s}{\alpha},$$

which gives $\alpha = 2$. Thus

$$\mathrm{osc}_\delta(t) \widetilde{P} \le (2c_1 + c_2) \delta^{1-s/2}.$$

(b) Now we want to show that

$$W = \int_a^b dt\, |\widetilde{P}(t+\delta) - \widetilde{P}(t-\delta)| \geq c\delta^{1-s/2}.$$

Set $a = 0, b = 2\pi$ for convenience. Then

$$\begin{aligned} W = & \int_0^{2\pi} dt \left| \sum_{c=1}^{\infty} q^{2c(s-2)} \sin q^c x \sum_{d=1}^{c} q^{-d(s-2)} \sin q^{c-d} x \,\cdot \right. \\ & \left. \cdot \left\{ \cos[(q^{2c} - q^{2(c-d)})(t+\delta)] - \cos[(q^{2c} - q^{2(c-d)})(t-\delta)] \right\} \right| \\ = & \int_0^{2\pi} dt \left| \sum_{c=1}^{\infty} q^{2c(s-2)} \sin q^c x \sum_{d=1}^{c} q^{-d(s-2)} \sin q^{c-d} x \,\cdot \right. \\ & \left. \cdot \left\{ -2 \sin[(q^{2c} - q^{2(c-d)})t] \sin[(q^{2c} - q^{2(c-d)})\delta] \right\} \right| . \end{aligned}$$

Using our standard arguments we multiply the integrand by a suitable function smaller or equal to 1:

$$\begin{aligned} W & \geq \int_0^{2\pi} dt\, |h(t)| |\widetilde{P}(t+\delta) - \widetilde{P}(t-\delta)| \\ & \geq \left| \int_0^{2\pi} dt\, h(t) [\widetilde{P}(t+\delta) - \widetilde{P}(t-\delta)] \right| . \end{aligned}$$

We choose $h(t) = \sin[(q^{2c} - q^{2(c-d)})t]$ and set $\delta = q^{-2N}$. It follows that

$$W \geq 2\pi q^{(2c-d)(s-2)} \left| \sin(q^c x) \sin(q^{c-d} x) \sin[(q^{2c} - q^{2(c-d)}) q^{-2N}] \right| .$$

We now want to show that for almost all x

$$W \geq c_3 \delta^{1-s/2} = c_3 q^{N(s-2).}$$

Set $2c - d = N$. Then

$$W \geq 2\pi q^{N(s-2)} \left| \sin(q^c x) \sin(q^{N-c} x) \sin[q^{2(c-N)} - q^{-2c}] \right| . \tag{39}$$

Thus it is enough to bound

$$\left| \sin(q^c x) \sin(q^{N-c} x) \sin[q^{2(c-N)} - q^{-2c}] \right| \tag{40}$$

from below.
Choose rational x/π. All the rational numbers in a given basis q have one of the two possible forms: finite or periodic. In the first case ($x/\pi = k/q^l$) we cannot find the lower bound on (40). We cannot succeed, because at these points the function $P_x(t)$ is smooth (cf. the next point of the proof). The other case means that x/π can be written as

$$x/\pi = 0.a_1 a_2 \ldots a_K (a_{K+1} \ldots a_{K+T}),$$

where again $(a_{K+1}\dots a_{K+T})$ denotes the periodic part. Therefore, for every $N>K$, $q^n x \bmod \pi$ can take only one of T values: $q^{K+1}x \bmod \pi, \dots, q^{K+T}x \bmod \pi$. Let us take $c=1$, $N>K$. Then $|\sin(q^c x)| = |\sin(qx)| > 0$ and is a constant. $|\sin(q^{N-1}x)|$ takes one of T values, none of which is 0, therefore it is always bounded from below by

$$\inf_{l=1,2,\dots,T} |\sin(q^{K+l}x)| > 0.$$

Also the last term can be bounded:

$$|\sin(q^{-2(N-c)} - q^{-2c})| \geq \sin(q^{-2} - q^{-2(N-1)}) \geq \sin q^{-3}$$

for $N \geq 3$. Thus for rational x/π with periodic expansion in q

$$W \geq c_3 q^{N(s-2)},$$

where $c_3 = 2\pi|\sin(qx)\sin(q^{-3}x)| \inf_{l=1,2,\dots,T}|\sin(q^{K+l}x)|$.
Consider now irrational x/π. Inequality (39) for $c=N$ takes form

$$\begin{aligned} W &\geq 2\pi q^{N(s-2)} \left|\sin(q^N x)\sin x \sin[1-q^{-2N}]\right| \\ &\geq c q^{N(s-2)} \left|\sin(q^N x)\right|, \end{aligned} \tag{41}$$

for $N \geq 2$. Instead of showing it can be bounded from below we will use it to prove that for almost every x

$$\dim_B \mathrm{graph} P_x(t) \geq 1 + s/2. \tag{42}$$

Let

$$x_n := q^n (x/\pi) \bmod 1. \tag{43}$$

Let

$$F_N^\alpha := \left\{ x \,:\, \exists n \geq N \quad \left(x_n \leq \frac{1}{q^{N\alpha}}\right) \vee \left(1 - x_n \leq \frac{1}{q^{N\alpha}}\right)\right\},$$

$\alpha \in [0,1]$. Let

$$F_\infty^\alpha := \bigcap_{N=1}^{\infty} F_N^\alpha$$

Clearly,

$$F_N^\alpha \supset F_{N+1}^\alpha \supset F_{N+2}^\alpha \dots$$

Since Renyi map (43) preserves the Lebesgue measure we have

$$\mu(F_N^\alpha) \leq 2\left(\frac{1}{q^N\alpha} + \frac{1}{q^{N+1}\alpha} + \frac{1}{q^{N+2}\alpha} + \dots\right) = \frac{2q}{q-1}\frac{1}{q^{N\alpha}}. \tag{44}$$

Therefore

$$0 \leq \mu(F_\infty^\alpha) \leq \inf_N \mu(F_N^\alpha) = 0 \tag{45}$$

It follows that for almost every $\{x_n\}$

$$\lim_{n\to\infty} \frac{\ln|\sin x_n|}{n} \geq \lim_{n\to\infty} \frac{\ln q^{-n\alpha}}{n} \geq \lim_{n\to\infty} \frac{q^{-n\alpha}}{2n} \geq -\alpha \ln q. \tag{46}$$

Thus for every $\alpha > 0$ and for almost every $x/\pi \in [0,1]$ we have

$$\dim_B \mathrm{graph} P_x(t) = \lim \left(2 - \frac{\ln \mathrm{Var}_\delta P_x(t)}{\ln q^{-2N}} \right) \tag{47}$$

$$\geq 2 + \lim \left(\frac{\ln \mathrm{Var}_\delta P_x(t)}{2N \ln q} \right) \tag{48}$$

But

$$\mathrm{Var}_\delta P_x(t) \geq W \tag{49}$$

therefore from (41)

$$\dim_B \mathrm{graph} P_x(t) \geq 2 + \lim \left(\frac{\ln c + N(s-2)\ln q + \ln|\sin(x_n \pi)|}{2N \ln q} \right) \tag{50}$$

$$= 1 + s/2 + \lim \left(\frac{\ln|\sin(x_n\pi)|}{2N \ln q} \right) \tag{51}$$

$$\geq 1 + s/2 - \alpha. \tag{52}$$

But α is arbitrary, thus

$$\dim_B \mathrm{graph} P_x(t) \geq 1 + s/2. \tag{53}$$

4. *For a discrete, dense set of points x_d, $D_t(x_d) = \dim_B \mathrm{graph} P_{x_d}(t) = 1$.*

   Let $x_{k,m} = \frac{m\pi}{q^k}$, where $k \in \mathbb{N}$, $m = 0, 1, \ldots, q^k - 1$. The set $\{x_{k,m}\}$ is dense in $[0,1]$. At these points $\Psi(x_{k,m}, t)$ is a sum of a finite number of terms:

$$\Psi\left(\frac{m\pi}{q^k}, t\right) = \sqrt{\frac{2}{\pi}\left(1 - \frac{1}{q^{2(2-s)}}\right)} \sum_{n=0}^{k-1} q^{(s-2)n} \sin(q^{n-k} m\pi)\, e^{-iq^{2n}t}.$$

   Therefore,

$$\dim_B \mathrm{graph} \left| \Psi\left(\frac{m\pi}{q^k}, t\right)\right|^2 = 1.$$

5. *For even q the average velocity $\frac{d\langle x\rangle}{dt}(t)$ is fractal with the dimension of its graph equal to $D_v = \max\{(1+s)/2, 1\}$.*

   Heuristically, this is rather obvious because

$$\frac{d\langle x\rangle}{dt}(t) \approx \sum_{k=1}^{\infty} \frac{q^{k(s-1)}}{q^{2k}} \sin q^{2k} t = \sum_{k=1}^{\infty} q^{2k(s-3)/2} \sin q^{2k} t.$$

   Thus the average velocity is essentially a Weierstrass-like function and the dimension of its graph should be

$$2 - \frac{3-s}{2} = \frac{1+s}{2}.$$

   It is enough to consider

$$W(t) := \sum_{k=1}^{\infty} \frac{q^{k(s-1)}}{q^{2k}-1} \sin(q^{2k}-1)t.$$

(a) Let

$$W_n(t) := \sum_{k=1}^{n} \frac{q^{k(s-1)}}{q^{2k}-1} \sin(q^{2k}-1)t.$$

Set $\delta = q^{-\alpha n}$. Then

$$|W_n'(t)| = \left|\sum_{k=1}^{n} q^{k(s-1)} \cos(q^{2k}-1)t\right| \leq \sum_{k=1}^{n} q^{k(s-1)} \leq c_1 \delta^{(1-s)/\alpha},$$

where

$$c_1 = \frac{q^{s-1}}{q^{s-1}-1}.$$

Therefore

$$\mathrm{osc}_\delta(t) W_n \leq 2c_1 \delta^{(1-s)/\alpha}\delta = 2c_1 \delta^{1+(1-s)/\alpha}.$$

Now, for

$$P_n(t) := W(t) - W_n(t)$$

we have

$$\begin{aligned} |P_n(t)| &= \left|\sum_{k=n+1}^{\infty} \frac{q^{k(s-1)}}{q^{2k}-1} \sin(q^{2k}-1)t\right| \leq \sum_{k=n+1}^{\infty} \frac{q^{k(s-1)}}{q^{2k}-1} \\ &\leq 2\sum_{k=n+1}^{\infty} \frac{q^{k(s-1)}}{q^{2k}} = c_2 \delta^{-\frac{s-3}{\alpha}}, \end{aligned}$$

where

$$c_2 = \frac{2q^{s-3}}{1-q^{s-3}}.$$

Thus

$$\mathrm{osc}_\delta(t) P_n \leq 2c_2 \delta^{\frac{3-s}{\alpha}}.$$

To obtain consistent estimate for both P_n and W_n we must set

$$1 + \frac{1-s}{\alpha} = \frac{3-s}{\alpha},$$

thus $\alpha = 2$ and $\delta = q^{-2N}$. Therefore

$$\mathrm{osc}_\delta(t) W \leq \mathrm{osc}_\delta(t) W_n + \mathrm{osc}_\delta(t) P_n \leq 2(c_1 + c_2)\delta^{2-\frac{s+1}{2}}.$$

(b) Consider

$$\begin{aligned} &\int_a^b |W(t+\delta) - W(t-\delta)| dt \\ &= \int_a^b dt \left|\sum_{k=1}^{\infty} \frac{q^{k(s-1)}}{q^{2k}-1} \cos(q^{2k}-1)t \,\sin(q^{2k}-1)\delta\right| \\ &\geq \left|\int_a^b dt\, h(t) f_N(t)\right| - \sum_{k\neq N} \left|\int_a^b dt\, h(t) f_k(t)\right|, \end{aligned}$$

where

$$f_k(t) = \frac{q^{k(s-1)}}{q^{2k}-1}\cos(q^{2k}-1)t\ \sin(q^{2k}-1)\delta.$$

Let $h(t) = \cos(q^{2N}-1)t$, $\delta = q^{-2N}$. Then

$$\begin{aligned}
&\left|\int_a^b h(t) f_N(t) dt\right| \\
&= \frac{q^{N(s-1)}}{q^{2N}-1}\sin(1-q^{-2N})\int_a^b \cos^2(q^{2N}-1)t\,dt \\
&\geq \frac{q^{N(s-1)}}{q^{2N}}\sin\frac{\pi}{6}\int_a^b \cos^2(q^{2N}-1)t\,dt \\
&\geq \frac{1}{2}q^{N(s-3)}\left[\frac{b-a}{2} + \frac{\sin(2b(q^{2N}-1)) - \sin(2a(q^{2N}-1))}{4(q^{2N}-1)}\right] \\
&\geq \frac{1}{2}\delta^{\frac{3-s}{2}}\left[\frac{b-a}{2} - \frac{2\cdot 2}{4\cdot q^{2N}}\right] \\
&= \frac{1}{2}\delta^{\frac{3-s}{2}}\left[\frac{b-a}{2} - \delta\right] \\
&\geq \frac{1}{8}\delta^{\frac{3-s}{2}}(b-a).
\end{aligned}$$

On the other hand

$$\begin{aligned}
&\left|\int_a^b h(t) f_k(t) dt\right| \\
&= \frac{q^{k(s-1)}}{q^{2k}-1}\sin\left[(q^{2k}-1)q^{-2N}\right]\int_a^b \cos(q^{2k}-1)t\cos(q^{2N}-1)t\,dt \\
&\leq \frac{q^{k(s-1)}}{q^{2k}-1}\left|\frac{\sin(b(q^{2N}-q^{2k})) - \sin(a(q^{2N}-q^{2k}))}{2(q^{2N}-q^{2k})} + \right. \\
&\qquad \left.\frac{\sin(b(q^{2N}+q^{2k})) - \sin(a(q^{2N}+q^{2k}))}{2(q^{2N}+q^{2k})}\right| \\
&\leq 2q^{k(s-3)}\left[\frac{1}{|q^{2N}-q^{2k}|} + \frac{1}{q^{2N}+q^{2k}}\right] \\
&\leq 2q^{k(s-3)}\left[\frac{1}{q^{2N}-q^{2(N-1)}} + \frac{1}{q^{2N}}\right] \\
&\leq 5\cdot q^{k(s-3)}\delta.
\end{aligned}$$

Therefore

$$\begin{aligned}
W &\geq \frac{1}{8}\delta^{(3-s)/2}(b-a) - \sum_k 5q^{k(s-3)}\delta \\
&\geq \frac{1}{8}\delta^{(3-s)/2}(b-a) - 5\frac{q^{s-3}}{q^{s-3}-1}\delta.
\end{aligned}$$

But $\frac{3-s}{2} < 1$, thus for large enough N (small enough δ) the first term dominates the other, therefore

$$W \geq c\delta^{2-(1+s)/2},$$

with $c = \frac{b-a}{16}$, for example.

From theorem 2 it follows that

$$D_v = \frac{1+s}{2}.$$

6. *The surface $P(x,t)$ has dimension $D_{xy} = 2 + s/2$.*
Setting x or t constant we have shown that oscillations are bounded by $c\delta^H$ where exponent H is one of $1, s, s/2$. We also showed the lower bound of variation is always $c\delta^H$ again with H being one of $1, s, s/2$. What's more, there is a dense set of points x for which $\mathrm{Var}_\delta P_x(t) \geq c\delta^{s/2}$. One can take for instance all rational x/π with periodic q-expansion. Thus from theorem 3 we have

$$D_{xt} = 1 + \max\{D_x, D_t\} = 2 + \frac{s}{2}. \tag{54}$$

□

A. Appendix: Auxiliary calculations

A.1. Probability density. Take the fractal wave function (8)

$$\Psi(x,t) = \sqrt{\frac{2}{\pi}\left(1 - q^{2(s-2)}\right)} \sum_{n=0}^{\infty} q^{n(s-2)} \sin(q^n x) e^{-iq^{2n}t}.$$

Let us calculate two useful forms of the probability density $P(x,t)$

$$\begin{aligned} P(x,t) &= |\Psi(x,t)|^2 \\ &= \frac{2}{\pi}\left(1 - q^{2(s-2)}\right) \sum_{m,n=0}^{\infty} q^{(m+n)(s-2)} \sin q^n x \sin q^m x \, e^{-i(q^{2n}-q^{2m})t}. \end{aligned}$$

Taking $k = m+n, l = n$ we obtain

$$\begin{aligned} P(x,t) &= \frac{2}{\pi}\left(1 - q^{2(s-2)}\right) \sum_{k=0}^{\infty} q^{k(s-2)} \sum_{l=0}^{k} \sin q^l x \sin q^{k-l} x \, e^{-i(q^{2l}-q^{2(k-l)})t}, \\ &= \frac{2}{\pi}\left(1 - q^{2(s-2)}\right) \sum_{k=0}^{\infty} q^{k(s-2)} \sum_{l=0}^{k} \sin q^l x \sin q^{k-l} x \cos(q^{2l} - q^{2(k-l)})t, \end{aligned} \tag{55}$$

Substitute $c = m$, $d = m - n$

$$\begin{aligned} P(x,t) \;=\; & \frac{2}{\pi}\left(1 - q^{2(s-2)}\right) \sum_{m=0}^{\infty} \Big\{ q^{2m(s-2)} \sin^2 q^m x + \\ & +2 \sum_{n<m} q^{(m+n)(s-2)} \sin q^n x \sin q^m x \cos(q^{2m} - q^{2n})t, \Big\} \end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{\pi}\left(1-q^{2(s-2)}\right)\left\{\sum_{m=0}^{\infty} q^{2m(s-2)}\sin^2 q^m x+\right.\\
&\quad\left.+2\sum_{c=1}^{\infty}\sum_{d=1}^{c} q^{(2c-d)(s-2)}\sin q^c x\sin q^{c-d}x\cos(q^{2c}-q^{2(c-d)})t\right\}\\
&= \frac{2}{\pi}\left(1-q^{2(s-2)}\right)\left\{\sum_{m=0}^{\infty} q^{2m(s-2)}\sin^2 q^m x+\right.\\
&\quad+2\sum_{c=1}^{\infty} q^{2c(s-2)}\sin q^c x\,\cdot\\
&\quad\left.\cdot\sum_{d=1}^{c} q^{-d(s-2)}\sin q^{c-d}x\cos\left[(q^2-1)\cdot q^{2(c-d)}\sum_{a=0}^{d-1}q^{2a}\right]t\right\}\\
&= \frac{2}{\pi}\left(1-q^{2(s-2)}\right)\sum_{m=0}^{\infty} q^{2m(s-2)}\sin^2 q^m x+\\
&\quad+\frac{4}{\pi}\left(1-q^{2(s-2)}\right)\sum_{c=1}^{\infty} q^{2c(s-2)}\sin q^c x\,\cdot\\
&\quad\cdot\sum_{d=1}^{c} q^{-d(s-2)}\sin q^{c-d}x\cos(q^2-1)(q^{2(c-1)}+\ldots+q^{2(c-d)})t \qquad (56)\\
&=: P_x(x)+P_{xt}(x,t).
\end{aligned}
$$

Note that the time-independent part

$$
\begin{aligned}
P_x(x) &= \frac{2}{\pi}\left(1-q^{2(s-2)}\right)\sum_{m=0}^{\infty} q^{2m(s-2)}\frac{1-\cos q^m 2x}{2}\\
&= \frac{1}{\pi}-\frac{\left(1-q^{2(s-2)}\right)}{\pi}\sum_{m=0}^{\infty} q^{m(2s-4)}\cos q^m 2x,
\end{aligned}
$$

is a Weierstrass-like function with the dimension $s' = \max\{2s-2, 1\} \in [1, 2)$ (i.e. for $s \in [1, 3/2]$, $s' = 1$). From the equation (56) one immediately gets the spectrum of $P(x,t)$: all the frequencies governing the time evolution are

$$
\omega_{c,d} = (q^2-1)(q^{2(c-1)}+\ldots+q^{2(c-d)}),
$$

where $c = 1, 2, \ldots$, $d = 1, 2, \ldots, c$. Thus all the frequencies divide by q^2-1 which is also the smallest frequency, so the fundamental period of $P(x,t)$ is $2\pi/(q^2-1)$.

A.2. Average velocity. Let us study the behavior of $\langle x\rangle$.

$$
\begin{aligned}
\langle x\rangle &= \int_0^{\pi} dx\, x\, |\Psi|^2\\
&= \frac{\pi}{2}-\frac{16}{\pi}\left(1-q^{2(s-2)}\right)\sum_{k=1}^{\infty}\frac{q^{k(s-1)}}{(q^{2k}-1)^2}\cos(q^{2k}-1)t.
\end{aligned}
$$

The above expression is valid only for even q. For odd q we have just the first term, which is $\pi/2$.

The average $x(t)$ is of class $\mathcal{C}^1$, because its derivative is given by an absolutely convergent series:

$$\begin{aligned}\left|\frac{d\langle x\rangle}{dt}\right| &= \left|\frac{16}{\pi}\left(1-q^{2(s-2)}\right)\sum_{k=1}^{\infty}\frac{q^{k(s-1)}}{q^{2k}-1}\sin(q^{2k}-1)t\right| \\ &\leq c\sum_{k=1}^{\infty}2\frac{q^{k(s-1)}}{q^{2k}} \\ &= 2c\frac{q^{s-3}}{1-q^{s-3}}. \end{aligned} \tag{57}$$

We showed in section (3) that (57) is fractal, while for odd q the average velocity $|d\langle x\rangle/dt|$, of course, is not. This seemingly strange behavior is caused by the fact that

$$\int_0^{\pi} dx \sin nx \sin mx$$

is non-zero only for m, n of different parity. However, if one slightly disturbs our function, for instance by changing an arbitrary number of terms to the next higher or lower eigenstates, the dimensions D_x and D_t will not be altered, but the average velocity will become fractal. In other words, with probability one, independently of the parity of q, the average velocity of the wave function

$$\Phi_0(x,t) = M_0\sum_{n=1}^{\infty}q^{n(s-2)}\sin((q^n\pm 1)x)e^{-i(q^n\pm 1)^2 t}. \tag{58}$$

is fractal characterized by the same dimensions D_x and D_t as the function currently studied.

An explicit example of a similar function for odd q it is

$$\Phi_1(x,t) = M_1\left[2^{s-2}\sin(2x)e^{-i2^2 t} + \sum_{n=1}^{\infty}q^{n(s-2)}\sin(q^n x)e^{-iq^{2n}t}\right]. \tag{59}$$

One can see the only difference between this example and the original one (8) is in the *first* term. This difference accounts for the smoothness or roughness of the average velocity. It is very interesting because normally one expects that it is the *asymptotic* behavior that determines the fractal dimension. Here we have exactly opposite case: a change in the first term (varying most slowly) of a series changes the dimension of a complicated function $\langle v\rangle$.

Average velocity of the wave packet (59) is smooth for even q and fractal for odd q. A function which gives fractal average velocity for both even and odd q is

$$\begin{aligned}\Phi_2(x,t) &= M_2\left[2^{s-2}\sin(2x)e^{-i2^2 t} + \sum_{n=0}^{\infty}q^{n(s-2)}\sin(q^n x)e^{-iq^{2n}t}\right] \\ &= M_2\left[2^{s-2}\sin(2x)e^{-i2^2 t} + \frac{1}{N}\Psi(x,t)\right],\end{aligned}$$

where M_2 is the normalization constant. On the other hand,

$$\Phi_3(x,t) = M_3 \left[\sum_{n=1}^{\infty} q^{n(s-2)} \sin(q^n x) e^{-iq^{2n}t} \right]$$

gives smooth average velocity for both even and odd q.

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